Dimensions of Biquadratic Spline Spaces over T-meshes

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Abstract *This paper discusses the dimensions of the spline spaces over T-meshes with lower degree. Two new concepts are proposed: extension of T-meshes and spline spaces with homogeneous boundary conditions. In the dimension analysis, the key strategy is linear space embedding with the operator of mixed partial derivative. The dimension of the original space equals the difference between the dimension of the image space and the rank of the constraints which ensuring any element in the image space has a preimage in the original space. Then the dimension formula and basis function construction of bilinear spline spaces of smoothness order zero over T-meshes are discussed in detail, and a dimension lower bound of biquadratic spline spaces over general T-meshes is provided. Furthermore, using level structure of hierarchical T-meshes, a dimension formula of biquadratic spline space over hierarchical T-meshes are proved. A topological explantation of the dimension formula is shown as well.*

1 Introduction

A T-mesh is a rectangular grid that allows T-junctions. T-splines, a type of point-based splines defined over a T-mesh, were proposed by T. W. Sederberg in [8, 9], and have become an important tool in geometric modeling and surface reconstruction. For T-splines, T-mesh plays two roles: defining its parametric domain decomposition fashion and representing the topological structure of the control net of a T-spline surface.

According to its definition, a T-spline is a piecewise polynomial, instead of a single polynomial, within a cell of the T-mesh. This is incompatible with the standard defining fashion of classic splines. Recall that, given a spline knot sequence, a univariate spline can be defined, which is a single polynomial between any two neighboring knots. Hence in [3], some of the present authors with some other coauthors introduced the concept of spline spaces over T-meshes, where every function in the space is exactly a polynomial within each cell of the T-mesh. A dimension formula can be proved with the B-net method [3] and the smoothing cofactor method [5] for the spline space $\mathbf{S}(m, n, \alpha, \beta, \mathscr{T})$ as $m \geqslant 2\alpha + 1$ and $n \geqslant 2\beta + 1$. Then in [4] we provided an approach to define the basis functions of the bicubic splines with smoothness order one. The applications of the basis functions in surface fitting is explored as well.

According the dimension formula in [3], one can obtain the specified dimension formulae for some spline spaces with low degree as $\mathbf{S}(1, 1, 0, 0, \mathscr{T})$, $\mathbf{S}(2, 2, 0, 0, \mathscr{T})$, $\mathbf{S}(3, 3, 0, 0, \mathscr{T})$, and $\mathbf{S}(3, 3, 1, 1, \mathscr{T})$. Furthermore, with similar approaches in [4, 7], one can construct their basis functions with some "good" properties, say, compact support, nonnegativity, and forming a partition of unity. In order to achieve high order smoothness with as low as possible degree, we expect to obtain the dimension formulae and basis functions construction for the spline spaces $\mathbf{S}(m, n, m-1, n-1, \mathscr{T})$. Especially, the most interesting ones are those for $\mathbf{S}(2, 2, 1, 1, \mathscr{T})$ and $\mathbf{S}(3, 3, 2, 2, \mathscr{T})$. In the paper, we only focus on the dimension of the former space.

In the following, we first introduce two concepts: the spline space with homogenous boundary conditions (HBC) and the extended T-meshes associated with some spline space. As a foundation of the

later analysis, we discuss in detail the dimension formula and basis functions construction for the space $\mathbf{S}(1,1,0,0,\mathscr{T})$. In [3] we have shown that the dimension of a bilinear spline space is the sum of the numbers of crossing vertices and boundary vertices in the given T-mesh. However, the proof proposed here shares the same fashion as in the analysis of a lower bound of dimensions of biquadratic spline spaces, and avoids the problem of recycling dependence of T-vertices.

An important technique in the dimension analysis is linear space embedding by an operator of mixed partial derivative, which embeds the space $\mathbf{S}(m,n,m-1,n-1,\mathscr{T})$ into the space $\mathbf{S}(m-1,n-1,m-2,n-2,\mathscr{T})$. A necessary and sufficient condition for describing any element in $\mathbf{S}(m-1,n-1,m-2,n-2,\mathscr{T})$ is the image of an element in $\mathbf{S}(m,n,m-1,n-1,\mathscr{T})$ by the operator. In the paper, we just discuss cases of $m=n=1$ or 2. With the method, a lower bound of dimensions of $\mathbf{S}(2,2,1,1,\mathscr{T})$ is proved. Finally, by making use of the level structure of hierarchical T-meshes, a dimension formula is provided for biquadratic spline spaces over hierarchical T-meshes.

The paper is organized as follows. In Section 2, the spline spaces over T-meshes is reviewed and two concepts are proposed: extension of T-meshes and spline spaces with homogeneous boundary conditions. Then the dimension formula and basis function construction of bilinear spline spaces of smoothness order zero over T-meshes are discussed in detail in Section 3, and a dimension lower bound of biquadratic spline spaces over general T-meshes is provided in Section. In Section 5, using level structure of hierarchical T-meshes, a dimension formula of biquadratic spline space over hierarchical T-meshes are proved. A topological explantation of the dimension formula is shown as well. Section 6 concludes the paper with some discussions.

2 T-meshes and Spline Spaces

A **T-mesh** is basically a rectangular grid that allows T-junctions. As for details of T-meshes, please refer to [3]. Here the T-meshes we discuss are regular, i.e., the domaines occupied by the T-meshes are rectangles. We adopt the compatible definitions of vertex, edge, cell with those in [3].

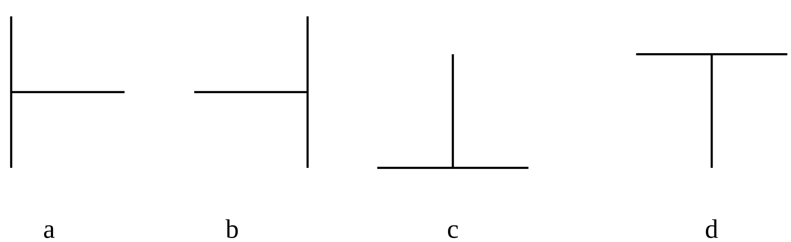


Figure 1: Horizontal T-vertices (a and b) and vertical T-vertices (c and d)

T-vertices in a T-mesh can be classified into two types: **horizontal T-vertices** and **vertical T-vertices**. The T-vertices as shown in Figures 1.a and b are horizontal ones, and those as shown in Figures 1.c and d are vertical ones. In [3], edges and c-edges are defined. Here we introduce a new type of edges, l-edges. A **horizontal/vertical l-edge** is a continuous line segment which consists of some horizontal/vertical interior edges and whose two endpoints are boundary vertices or horizontal/vertical T-vertices. In other words, an l-edge is a longest possible line segment in the mesh. The boundary of a regular T-mesh consists of four l-edges, which are called boundary l-edges. The other l-edges in the T-mesh are called interior l-edges. For example, in Figure 2, the given T-mesh $\mathscr{T}$ has three horizontal interior l-edges and three vertical interior l-edges, respectively.

Given two series of real numbers x_i, $i=1,\ldots,m$, and y_j, $j=1,\ldots,n$, where $x_i < x_{i+1}$ and $y_j < y_{j+1}$, a rectangular grid can be formed with vertices (x_i,y_j), $i=1,\ldots,m$, $j=1,\ldots,n$. This

grid is called a **tensor-product mesh**, denoted by $(x_1, \ldots, x_m) \times (y_1, \ldots, y_n)$, which is a special type of T-mesh. From a regular T-mesh $\mathscr{T}$, a tensor-product mesh $\mathscr{T}^c$ can be constructed by extending all the interior l-edges to the boundary. $\mathscr{T}^c$ is called the **associated tensor-product mesh** with $\mathscr{T}$. See Figure 2 for an example.

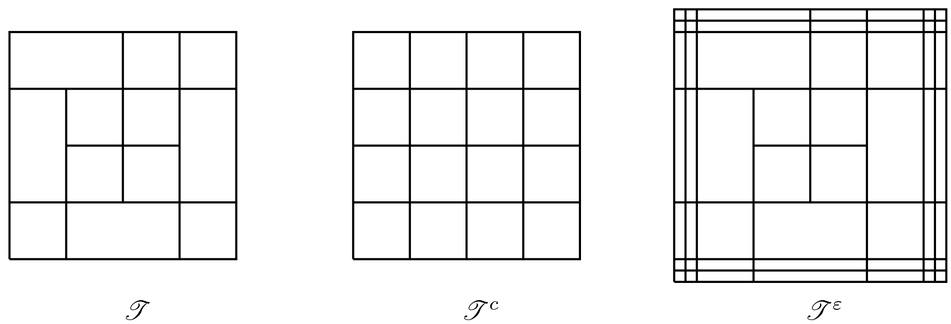


Figure 2: A T-mesh with its associated tensor-product mesh $\mathscr{T}^c$ and its extension $\mathscr{T}^\varepsilon$.

2.1 Spline spaces over T-meshes

Given a T-mesh $\mathscr{T}$, we use $\mathscr{F}$ to denote all the cells in $\mathscr{T}$ and Ω to denote the region occupied by all the cells in $\mathscr{T}$. In [3], the following spline space definition is proposed:

$$\mathbf{S}(m, n, \alpha, \beta, \mathscr{T}) := \{f(x, y) \in C^{\alpha,\beta}(\Omega) : f(x, y)|_\phi \in \mathbb{P}_{mn}, \forall \phi \in \mathscr{F}\}, \tag{2.1}$$

where $\mathbb{P}_{mn}$ is the space of all the polynomials with bi-degree (m, n), and $C^{\alpha,\beta}$ is the space consisting of all the bivariate functions which are continuous in Ω with order α along x direction and with order β along y direction. It is obvious that $\mathbf{S}(m, n, \alpha, \beta, \mathscr{T})$ is a linear space.

Now we introduce a new spline space over T-meshes with the following defintion:

$$\overline{\mathbf{S}}(m, n, \alpha, \beta, \mathscr{T}) := \{f(x, y) \in C^{\alpha,\beta}(\mathbb{R}^2) : f(x, y)|_\phi \in \mathbb{P}_{mn}, \forall \phi \in \mathscr{F}, \text{ and } f|_{\mathbb{R}^2\setminus\Omega} \equiv 0\}, \tag{2.2}$$

which is called a spline space over the given T-mesh $\mathscr{T}$ with **homogeneous boundary conditions (HBC)**.

2.2 Extended T-meshes

In order to discuss the dimension of the spline space $\mathbf{S}(m, n, m-1, n-1, \mathscr{T})$, we extend the given regular T-mesh $\mathscr{T}$ in the following fashion: Pickup a tensor-product mesh $\mathscr{M}$ with $2(m+1)$ vertical lines and $2(n+1)$ horizontal lines, such that the central rectangle of $\mathscr{M}$ is the same size as Ω, the region occupied by $\mathscr{T}$; Then all the edges with an endpoint on the boundary of $\mathscr{T}$ are extended to reach the boundary of $\mathscr{M}$. The result mesh, denoted as $\mathscr{T}^\varepsilon$, is called an extension of the original T-mesh (or say, an extended T-mesh) associated with the present spline space. Figure 2 provides an example, where the T-mesh is extended associated with $\mathbf{S}(2, 2, 1, 1, \mathscr{T})$.

It should be noted that the extension is associated with a spline space. Hence, associated with different spline spaces, one will obtain different extended T-meshes. The following theorem shows that, using an extension of T-mesh, the dimension analysis of $\overline{\mathbf{S}}(2, 2, 1, 1, \mathscr{T}^\varepsilon)$ is the same as the dimension analysis of $\mathbf{S}(2, 2, 1, 1, \mathscr{T})$.

Theorem 2.1 *Given a T-mesh $\mathscr{T}$, assume $\mathscr{T}^\varepsilon$ is its extension associated with $\mathbf{S}(2,2,1,1,\mathscr{T})$. Then*

$$\mathbf{S}(2,2,1,1,\mathscr{T}) = \overline{\mathbf{S}}(2,2,1,1,\mathscr{T}^\varepsilon)|_{\mathscr{T}}, \tag{2.3}$$

$$\dim \mathbf{S}(2,2,1,1,\mathscr{T}) = \dim \overline{\mathbf{S}}(2,2,1,1,\mathscr{T}^\varepsilon). \tag{2.4}$$

Proof: We first prove Equation (2.3). Since $\mathscr{T}^\varepsilon$ is an extension of $\mathscr{T}$, it follows that

$$\overline{\mathbf{S}}(2,2,1,1,\mathscr{T}^\varepsilon)|_{\mathscr{T}} \subset \mathbf{S}(2,2,1,1,\mathscr{T}).$$

In the following we will prove that, for any $f \in \mathbf{S}(2,2,1,1,\mathscr{T})$, there exists $\bar{f} \in \overline{\mathbf{S}}(2,2,1,1,\mathscr{T}^\varepsilon)$ such that $\bar{f}|_{\mathscr{T}} = f$.

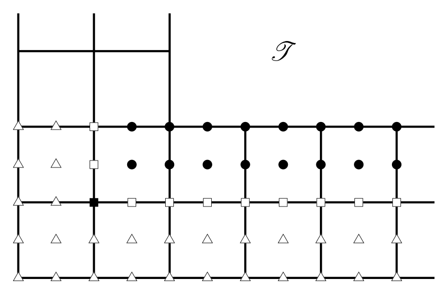


Figure 3: Bézier ordinates in an extended T-mesh

Here we define the function $\bar{f}$ over $\mathscr{T}^\varepsilon$ by assigning its Bézier ordinates in every cell of $\mathscr{T}^\varepsilon \backslash \mathscr{T}$. Since $\bar{f} \in C^{1,1}$, and there does not exist any interior T-vertices in $\mathscr{T}^\varepsilon \backslash \mathscr{T}$, $\bar{f}$ shares the same ordinates along the common boundary between two neighboring cells. (Therefore, these same ordinates appears once in Figure 3, which illustrates the distribution of the Bézier ordinates in the left-bottom part of the extended region $\mathscr{T}^\varepsilon \backslash \mathscr{T}$.) According to the specification of f over $\mathscr{T}$, it follows that the ordinates labeled with "•" are determined. Since $\bar{f}$ meets the zero function along the boundary of the extended T-mesh $\mathscr{T}^\varepsilon$ with order one, the ordinates labeled with "△" are determined as well. The rest ordinates labeled with "□" can be determined by its corresponding neighboring horizontal or vertical ordinates labeled with "△" and "•". Here the continuous condition is the collinear of the corresponding three points. It should be noted that the ordinates themselves determined in this fashion ensures that they define a continuous spline automatically.

After determining the former ordinates, the ordinate labeled with "■" can be determined as follows: It follows that the eight ordinates neighboring the ordinate "■" define a bilinear function. When the ordinate corresponding to "■" also lies on the bilinear function, the continuous of order one along two directions can be guaranteed. In fact, when dealing with the determination of all the ordinates in the extension region around $\mathscr{T}$, if we meet cases of recycling determinations, we can address the ordinates using this method of bilinear functions construction.

Hence we obtain the specification of the function $\bar{f}$ in $\mathbf{S}(2,2,1,1,\mathscr{T}^\varepsilon)$, which satisfies $\bar{f}|_{\mathscr{T}} = f$. According to the former analysis, it follows that

$$\mathbf{S}(2,2,1,1,\mathscr{T}) = \overline{\mathbf{S}}(2,2,1,1,\mathscr{T}^\varepsilon)|_{\mathscr{T}}.$$

In order to prove Equation (2.4), we will prove that, for any nonzero function $\bar{f} \in \overline{\mathbf{S}}(2,2,1,1,\mathscr{T}^\varepsilon)$, it follows that $\bar{f}|_{\mathscr{T}} \not\equiv 0$. Suppose $\bar{f}|_{\mathscr{T}} \equiv 0$, i.e., there exists $p = (x_0, y_0) \in \mathscr{T}^\varepsilon \backslash \mathscr{T}$ such that $\bar{f}(x_0, y_0) \neq 0$. And assume $\Omega = (x_l, x_r) \times (y_b, y_t)$. If $x_l \leqslant x_0 \leqslant x_r$, $y_0 < y_b$, then $\bar{f}(x_0, y)$, $y \leqslant y_b$ is a quadratic spline, whose support has just three breakpoints. This contradicts with the fact about the minimal support

of quadratic splines (The support of a nonzero quadratic spline should have at least four breakpoints). Similarly, we can show that p is outside of the regions $x_l \leqslant x_0 \leqslant x_r$, $y_0 > y_t$; $y_l \leqslant y_0 \leqslant y_r$, $x_0 < x_l$ and $y_b \leqslant y_0 \leqslant y_t$, $x_0 > x_r$. Therefore, without loss of generality, we assume $x_0 < x_r$, $y_0 < y_b$. But this also results in a nonzero quadratic spline with three breakpoints in its support. Hence we obtain $\bar{f}|_{\mathscr{T}} \not\equiv 0$, and has proved that

$$\dim \mathbf{S}(m,n,\alpha,\beta,\mathscr{T}) = \dim \overline{\mathbf{S}}(m,n,\alpha,\beta,\mathscr{T}^{\varepsilon}).$$

This completes the proof of the theorem. ∎

According to Theorem 2.1, we will consider only the biquadratic spline spaces over T-meshes with HBC.

Remark: A similar analysis can show that, for any m and n, it follows that

$$\mathbf{S}(m,n,m-1,n-1,\mathscr{T}) = \overline{\mathbf{S}}(m,n,m-1,n-1,\mathscr{T}^{\varepsilon})|_{\mathscr{T}}, \tag{2.5}$$

$$\dim \mathbf{S}(m,n,m-1,n-1,\mathscr{T}) = \dim \overline{\mathbf{S}}(m,n,m-1,n-1,\mathscr{T}^{\varepsilon}), \tag{2.6}$$

where $\mathscr{T}^{\varepsilon}$ is an extension of $\mathscr{T}$ associated with $\mathbf{S}(m,n,m-1,n-1,\mathscr{T})$. Hence one can replace the dimension discussion for $\mathbf{S}(m,n,m-1,n-1,\mathscr{T})$ with that for $\overline{\mathbf{S}}(m,n,m-1,n-1,\mathscr{T}^{\varepsilon})$.

3 Dimensions and Basis Functions of $\overline{\mathrm{S}}(1,1,0,0,\mathscr{T})$

In [3] we have proved the following dimension formula of the spline space $\mathbf{S}(m,n,\alpha,\beta,\mathscr{T})$ over T-mesh:

$$\begin{aligned}\dim \mathbf{S}(m,n,\alpha,\beta,\mathscr{T}) = {} & F(m+1)(n+1) - E_h(m+1)(\beta+1) \\ & -E_v(n+1)(\alpha+1) + V(\alpha+1)(\beta+1),\end{aligned} \tag{3.1}$$

where $m \geqslant 2\alpha+1$, $n \geqslant 2\beta+1$, F is the number of all the cells in $\mathscr{T}$, E_h and E_v the numbers of horizontal and vertical interior edges, respectively, and V the number of interior vertices (including crossing and T-vertices). Specially, as $m = n = 1$ and $\alpha = \beta = 0$, it follows that

$$\dim \mathbf{S}(1,1,0,0,\mathscr{T}) = V^{+} + V^{b}, \tag{3.2}$$

where V^{+} is the number of crossing vertices, and V^{b} the number of boundary vertices. Using an extension of the T-meshes associated with $\mathbf{S}(1,1,0,0,\mathscr{T})$, it follows that

$$\dim \overline{\mathbf{S}}(1,1,0,0,\mathscr{T}^{\varepsilon}) = \dim \mathbf{S}(1,1,0,0,\mathscr{T}) = V_{\varepsilon}^{+}, \tag{3.3}$$

where V_{ε}^{+} is the number of the crossing vertices in the extended T-mesh $\mathscr{T}^{\varepsilon}$, which equals the sum of the numbers of crossing vertices and boundary vertices in $\mathscr{T}$.

Now we will prove that the former formula holds for a general regular T-mesh (not just an extended T-mesh). The method taken in the proof is similar with the method proposed in the next section for proving the dimension properties of $\overline{\mathbf{S}}(2,2,1,1,\mathscr{T})$. This method solves the problem of recycling dependence of T-vertices which happens in the proof proposed in [3].

Now we introduce some notation and lemmas for the proof. In the given T-mesh $\mathscr{T}$, let E denote the number of interior l-edges, and V^{+} the number of crossing vertices. Define $\overline{\mathbf{S}}(0,0,-1,-1,\mathscr{T})$ to be the space consisting of functions which are a constant in each cell of the T-mesh $\mathscr{T}$, and have no smoothness requirement between neighboring cells. It is obvious that its dimension is the number of all the cells in the T-mesh.

3.1 Operator of mixed partial derivative

The operator of mixed partial derivative is introduced as follows:

$$\mathcal{D} := \frac{\partial^2}{\partial x \partial y} : \overline{\mathbf{S}}(m, n, m-1, n-1, \mathscr{T}) \to \overline{\mathbf{S}}(m-1, n-1, m-2, n-2, \mathscr{T}), \tag{3.4}$$

where $m, n \geqslant 1$. Since the function satisfies HBC, the operator is a one-to-one, but not onto mapping. For example, for any nonzero $g \in \overline{\mathbf{S}}(m, n, m-1, n-1, \mathscr{T})$, it follows that $\mathcal{D}(g)$ must reach positive values in some parts and negative values in some other parts. Hence a nonnegative function in $\overline{\mathbf{S}}(m-1, n-1, m-2, n-2, \mathscr{T})$ has no pre-image under the operator $\mathcal{D}$.

Define

$$\mathcal{I}(g)(x, y) = \int_{-\infty}^{x} \int_{-\infty}^{y} g(s, t) ds dt. \tag{3.5}$$

It follows that, for any $f \in \overline{\mathbf{S}}(m, n, m-1, n-1, \mathscr{T})$, $\mathcal{I} \circ \mathcal{D}(f) = f$. Hence $\mathcal{I}$ can be considered to be the inverse operator of $\mathcal{D}$.

It follows that, for any $g \in \overline{\mathbf{S}}(m-1, n-1, m-2, n-2, \mathscr{T})$, $\mathcal{I}(g)$ is a piecewise polynomial of degree (m, n) with smoothness $C^{m-1,n-1}$. With respect to functions in $\overline{\mathbf{S}}(m, n, m-1, n-1, \mathscr{T})$, $\partial^m \mathcal{I}(g)/\partial x^m$ or $\partial^n \mathcal{I}(g)/\partial y^n$ may be discontinuous inside some cells of $\mathcal{T}$. But the discontinuity must happen on the extension of some l-edges of $\mathcal{T}$. Let

$$d_{m,n} = \dim \overline{\mathbf{S}}(m, n, m-1, n-1, \mathscr{T}).$$

If we have known $d_{m-1,n-1}$, i.e., the dimension of $\overline{\mathbf{S}}(m-1, n-1, m-2, n-2, \mathscr{T})$, and the number $r_{m-1,n-1}$ of linear-independent constraints ensuring $\mathcal{I}(g) \in \overline{\mathbf{S}}(m, n, m-1, n-1, \mathscr{T})$ for any $g \in \overline{\mathbf{S}}(m-1, n-1, m-2, n-2, \mathscr{T})$, then it follows that

$$d_{m,n} = d_{m-1,n-1} - r_{m-1,n-1}. \tag{3.6}$$

On the other hand, if there are $r'_{m-1,n-1}$ constraints, which may be linear dependent, proposed for ensuring $\mathcal{I}(g) \in \overline{\mathbf{S}}(m, n, m-1, n-1, \mathscr{T})$ for any $g \in \overline{\mathbf{S}}(m-1, n-1, m-2, n-2, \mathscr{T})$, then it follows that

$$d_{m,n} \geqslant d_{m-1,n-1} - r'_{m-1,n-1}. \tag{3.7}$$

Equations (3.6) and (3.7) are used to prove the dimension formulae of bilinear and biquadratic spline spaces over T-meshes in the rest of the paper.

3.2 Some lemmas

As $m = n = 1$, the following lemma proposes the constraints ensuring $\mathcal{I}(g) \in \overline{\mathbf{S}}(1, 1, 0, 0, \mathscr{T})$ for any $g \in \overline{\mathbf{S}}(0, 0, -1, -1, \mathscr{T})$.

Lemma 3.1 *Given a regular T-mesh $\mathscr{T}$, let $g \in \overline{\mathbf{S}}(0, 0, -1, -1, \mathscr{T})$. Then*

$$\mathcal{I}(g)(x, y) \in \overline{\mathbf{S}}(1, 1, 0, 0, \mathscr{T})$$

if and only if the following two sets of conditions are satisfied simultaneously:

1. *For any horizontal l-edge l^h,*

$$\int_{x_0}^{x_1} g(s, y_0-) ds = \int_{x_0}^{x_1} g(s, y_0+) ds, \tag{3.8}$$

where the two end-points of l^h are with coordinates (x_0, y_0) and (x_1, y_0);

2. *For any vertical l-edge l^v,*

$$\int_{y_0}^{y_1} g(x_0-,t)dt = \int_{y_0}^{y_1} g(x_0+,t)dt, \tag{3.9}$$

where the two end-points of l^v are with coordinates (x_0, y_0) and (x_0, y_1).

Proof: For any horizontal l-edge l^h, its right end-point (x_1, y_0) is a T-vertex or a boundary vertex. Suppose the edge through the vertex (x_1, y_0) and perpendicular with l^h is e^v (as shown in Figure 4). If $\mathcal{I}(g)(x,y) \in \overline{\mathbf{S}}(1,1,0,0,\mathscr{T})$, then $\mathcal{I}(g)|_{e^v}$ is a linear polynomial in a neighborhood of (x_1, y_0). Hence it follows that

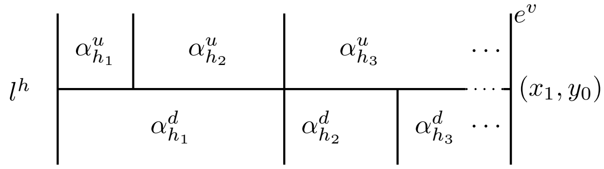


Figure 4: A horizontal l-edge (1)

$$\frac{\partial}{\partial y}\mathcal{I}(g)(x_1, y_0-) = \frac{\partial}{\partial y}\mathcal{I}(g)(x_1, y_0+). \tag{3.10}$$

According to the definition of $\mathcal{I}$ in Equation (3.5), Equation (3.10) can be rewritten into

$$\int_{-\infty}^{x_1} g(s, y_0-)ds = \int_{-\infty}^{x_1} g(s, y_0+)ds. \tag{3.11}$$

Extend l^h to reach the left boundary of the T-mesh. If there are not other horizontal l-edges on the extension edge, then, in every cell that intersects the extension edge, g is a pure linear polynomial. Then $g(s, y_0-) = g(s, y_0+)$ as $s \leqslant x_0$, and

$$\int_{-\infty}^{x_0} g(s, y_0-)ds = \int_{-\infty}^{x_0} g(s, y_0+)ds. \tag{3.12}$$

According to Equations (3.11) and (3.12), it follows that the equation (3.8) in the first condition holds for any horizontal l-edge.

It there exist some other horizontal l-edges on the extension edge, then we run through these horizontal l-edges from left to right. For the first one, we can prove that Equation (3.8) holds with the former approach. Then we consider the other l-edges one by one, and prove that the corresponding equation (3.12) holds as well, and then Equation (3.8) holds. Hence we finish to prove Equation (3.8) holds for every horizontal l-edge.

For any vertical l-edge, we can prove that the corresponding equation (3.9) holds in a similar way. Hence the necessity of the lemma is proved.

Now we will prove the sufficiency of the lemma. Suppose $g \in \overline{\mathbf{S}}(0,0,-1,-1,\mathscr{T})$ satisfies the two sets of conditions in the lemma, but $\mathcal{I}(g) \notin \overline{\mathbf{S}}(1,1,0,0,\mathscr{T})$, i.e., there exists a cell where $\partial\mathcal{I}(g)/\partial x$ or $\partial\mathcal{I}(g)/\partial y$ has at least a discontinuous point. Define two sets consisting of the cells of $\mathscr{T}$ as follows. Let $\mathcal{B}_x$ denote all the cells where $\partial\mathcal{I}(g)/\partial x$ has at least a discontinuous point; Let $\mathcal{B}_y$ denote all the cells where $\partial\mathcal{I}(g)/\partial y$ has at least a discontinuous point. Then according to the assumption, $\mathcal{B}_x \cup \mathcal{B}_y$ is not empty. Without loss of generality, we assume $\mathcal{B}_y$ is not empty. Consider a cell in $\mathcal{B}_y$ whose left-bottom corner has the minimal y coordinate in $\mathcal{B}_y$. If there exist more than one such cell, select one from them such that its left-bottom corner has the minimal x coordinate. Then now we have selected a unique cell c. In c, $\partial\mathcal{I}(g)/\partial y$ has at least one discontinuous point $(\bar{x}, \bar{y})$. Hence it follows that

$$\int_{-\infty}^{\bar{x}} g(s, \bar{y}-)ds \neq \int_{-\infty}^{\bar{x}} g(s, \bar{y}+)ds. \tag{3.13}$$

Consider the horizontal straight line through $(\bar{x},\bar{y})$. On the straight line if there does not exist an l-edge of $\mathscr{T}$ on the left side of $(\bar{x},\bar{y})$, then Equation (3.13) fails to hold. If there exists at least one l-edge on the left side of $(\bar{x},\bar{y})$, then according to the first condition, a contradiction arises as well. Hence $\mathcal{B}_y$ is empty. In the same fashion, it follows that $\mathcal{B}_x$ is empty as well. This contradicts with $\mathcal{B}_x \cup \mathcal{B}_y$ non-empty. Therefore we finish the proof of the sufficiency. ■

There are $E+4$ conditions in Lemma 3.1. The following lemma states that these $E+4$ conditions are equivalent with $E+2$ conditions in another form.

Lemma 3.2 *Given a regular T-mesh $\mathscr{T}$, the occupied rectangle by $\mathscr{T}$ is $(x_l,x_r)\times(y_b,y_t)$, where the different y coordinates of horizontal l-edges are $y_0<y_1<\ldots<y_n$. Suppose $g\in\overline{\mathbf{S}}(0,0,-1,-1,\mathscr{T})$. Then*

$$\int_{x_l}^{x_r} g(s,y_i-)ds = \int_{x_l}^{x_r} g(s,y_i+)ds, \quad i=0,1,\ldots,n \tag{3.14}$$

is equivalent with

$$\int_{x_l}^{x_r} g(s,y)ds = 0, \quad y\in(y_i,y_{i+1}), \quad i=0,\ldots,n-1. \tag{3.15}$$

It should be noted that, since g is a piecewise constant, $\int_{x_l}^{x_r} g(s,y)ds$, $y\in(y_i,y_{i+1})$, is a constant independent on y.

A similar statement can be made for vertical l-edges.

Proof: Since g is a piecewise constant in $\mathscr{T}$, and zero out of $\mathscr{T}$, it follows that

$$\int_{x_l}^{x_r} g(s,y_0-)ds = \int_{x_l}^{x_r} g(s,y)ds = 0, \quad y<y_0,$$
$$\int_{x_l}^{x_r} g(s,y_0+)ds = \int_{x_l}^{x_r} g(s,y)ds, \quad y_0<y<y_1.$$

Hence according to Equation (3.14) as $i=0$, we have

$$\int_{x_l}^{x_r} g(s,y)ds = 0, \quad y\in(y_0,y_1).$$

This means that Equation (3.15) holds as $i=0$. Then we consider cases as $i=1,2,\ldots,n-1$. Similar discussions guarantee Equation (3.15) holds as $i=1,2,\ldots,n-1$.

On the other hand, for any $i=0,1,2,\ldots,n-1$, if all the quations in (3.15) hold, it is easy to prove that all the equations in (3.14) hold. ■

Recall that, for a given T-mesh $\mathscr{T}$, an associated tensor product mesh $\mathscr{T}^c$ can be obtained by extending all the interior l-edges to the boundary of $\mathscr{T}$. Suppose there are E' interior l-edges in $\mathscr{T}^c$ and E interior l-edges in $\mathscr{T}$, where $E'\leqslant E$ (since it is possible that more than one l-edges in $\mathscr{T}$ on the same l-edge in $\mathscr{T}^c$). Select any horizontal l-edge l from $\mathscr{T}^c$, and suppose the horizontal l-edges $l_1,\ldots,l_k$ in $\mathscr{T}$ lie on l. Then it follows that the constraints

$$\int_{l_i} g(s,\bar{y}-)ds = \int_{l_i} g(s,\bar{y}+)ds, \quad i=1,\ldots,k$$

is equivalent with the constraints

$$\int_{l_i} g(s,\bar{y}-)ds = \int_{l_i} g(s,\bar{y}+)ds, \quad i=1,\ldots,k-1, \quad \int_{l} g(s,\bar{y}-)ds = \int_{l} g(s,\bar{y}+)ds,$$

where $\bar{y}$ is the vertical coordinate of l. According to Lemma 3.2, all the following constraints, with respect to any horizontal l-edge l in $\mathscr{T}^c$,

$$\int_{l} g(s,\bar{y}-)ds = \int_{l} g(s,\bar{y}+)ds$$

are equivalent with the integrations of g along all the span formed by two neighboring horizontal l-edges in $\mathscr{T}^c$ are zero. A similar equivalence can be made for vertical l-edges. Here we should noted that, for any boundary l-edge, there is only one constraint stated in Lemma 3.1. Hence the necessary and sufficient conditions that $\mathcal{I}(g) \in \overline{\mathbf{S}}(1,1,0,0,\mathscr{T})$ for $g \in \overline{\mathbf{S}}(0,0,-1,-1,\mathscr{T})$ are $E+2$ constraints in this fashion. On the other hand, if $\mathscr{T}^c = (x_0, x_1, \ldots, x_m) \times (y_0, y_1, \ldots, y_n)$, it follows that

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} g(s,t) ds dt = \sum_{i=0}^{m-1} (x_{i+1} - x_i) C_i = \sum_{j=0}^{n-1} (y_{j+1} - y_j) D_j,$$

where

$$C_i = \int_{y_0}^{y_n} g(x,y) dy, \quad x \in (x_i, x_{i+1}),$$
$$D_j = \int_{x_0}^{x_m} g(x,y) dx, \quad y \in (y_j, y_{j+1}).$$

Hence these $E+2$ constraints is with defective rank at least one. The latter dimension theorem 3.4 will show that the defective rank is exactly one.

Up to now, we have finished the description of the constraints that $\mathcal{I}(g) \in \overline{\mathbf{S}}(1,1,0,0,\mathscr{T})$. Before we state the dimension theorem, a topological equation of T-meshes is proposed in the following lemma.

Lemma 3.3 *Given a regular T-mesh $\mathscr{T}$, suppose $\mathscr{T}$ has F cells, V^+ crossing vertices, and E interior l-edges. Then*

$$F = V^+ + E + 1.$$

Proof: Suppose, in $\mathscr{T}$, there are V^T T-vertices, V^{bT} boundary vertices (excluding four corner points). Since every cell has four vertices, running through all the cells will meet every crossing vertices four times, every interior T-vertices twice, every corner points once, and every other boundary vertices twice. Hence it follows that

$$4F = 4V^+ + 2V^T + 2V^{bT} + 4.$$

On the other hand, the end-points of every interior l-edge are either interior T-vertices or boundary vertices (not corner points). Therefore, we have $V^T + V^{bT} = 2E$. From these two equations one gets $F = V^+ + E + 1$. ■

3.3 Dimension theorem

Theorem 3.4 *Given a regular T-mesh $\mathscr{T}$ with V^+ crossing vertices, it follows that*

$$\dim \overline{\mathbf{S}}(1,1,0,0,\mathscr{T}) = V^+.$$

Proof: We first prove that $\dim \overline{\mathbf{S}}(1,1,0,0,\mathscr{T}) \leqslant V^+$. Suppose a function $f \in \overline{\mathbf{S}}(1,1,0,0,\mathscr{T})$ reaches zero at all the crossing vertices. We intend to prove that $f \equiv 0$.

Suppose $f \not\equiv 0$. Without loss of generality, we assume f is greater than zero in some regions in $\mathscr{T}$. Then there exists a point p in $\mathscr{T}$ such that $f(p) = \delta = \max f > 0$. Let p be in the cell c. There will only happen the following two cases:

1. there exists an edge e of c such that f is a constant δ along e;

2. for any edge e of c, f is not a constant along e.

Consider Case 1. Let l_0 denote the l-edge on which e lies. Define a set L which consists of all the l-edges on which f are constants δ. P consists of all the end-points of the l-edges in L. Since $l_0 \in L$, both L and P are non-empty. Now we category the vertices in P into two types. If a vertex in P is also an interior vertex on another l-edge in L, then the vertex is called a flat vertex. Otherwise it is called a non-flat vertex. In the following we will prove that there must exist at least one non-flat vertex in P. If so, select one non-flat vertex q. Then q must lie on an l-edge l_1 which is not in L. q is an interior point on l_1. Since $f(q) = \delta$ and $f|_{l_1}$ is not a constant, there exists a point r on l_1 such that $f(r) > \delta$, which contracts with the fact that δ is the maximum of f over $\mathscr{T}$.

In fact, the non-flat vertex q can be selected to the vertex in P with the minimal y coordinate. If there exist more than one vertex with the minimal y coordinate, one with the minimal x coordinate among them is selected. Such the selection ensures that q is a non-flat vertex. If not, q lies on two l-edges l_2 and l_3 in L, where l_2 is horizontal and l_3 vertical. If q is a horizontal T-vertex, then the bottom end-point of l_3 has a smaller y coordinate than q. If q is a vertical T-vertex, then the left end-point of l_2 has the same y coordinate as q, but with smaller x coordinate than q. Both the cases contradict with the selection of q. Hence q is a non-flat point. Then we can prove $f \equiv 0$ for Case 1.

Now we consider Case 2. Since $f|_c$ is a bilinear function and f is not a constant along any edges of c, f reaches its maximum only at one of its corners. Hence p is a corner of c, which is a T-vertex, say a horizontal T-vertex. The vertical l-edge through p is assumed to be l_4, which takes p as its interior point. Since $f(p) = \delta$ and $f|_{l_4}$ is not a constant, there exists a point s on l_4 such that $f(s) > \delta$, which contracts with the fact that δ is the maximum of f over $\mathscr{T}$.

Summarizing the consideration for both the cases, it follows that $f \equiv 0$. Then all the crossing vertices in $\mathscr{T}$ form a determining set of the spline space $\overline{\mathbf{S}}(1,1,0,0,\mathscr{T})$. According to the theory of the determining sets in spline functions [1, 2], one gets

$$\dim \overline{\mathbf{S}}(1,1,0,0,\mathscr{T}) \leqslant V^+. \tag{3.16}$$

On the other hand, we will prove that $\dim \overline{\mathbf{S}}(1,1,0,0,\mathscr{T}) \geqslant V^+$ in the following. To do so, we consider the operator of mixed partial derivative defined in Equation (3.4) as $m = n = 1$:

$$\mathcal{D} : \overline{\mathbf{S}}(1,1,0,0,\mathscr{T}) \to \overline{\mathbf{S}}(0,0,-1,-1,\mathscr{T}).$$

Here $\mathcal{D}$ is injective. The spline space $\overline{\mathbf{S}}(0,0,-1,-1,\mathscr{T})$ is with dimension F, the number of the cells in $\mathscr{T}$. According to the analysis in the former section, in order to ensure $\mathcal{I}(g) \in \overline{\mathbf{S}}(1,1,0,0,\mathscr{T})$, one needs to satisfy $E+2$ constraints, which have defective rank at least one. Then according to Lemma 3.3,

$$\dim \overline{\mathbf{S}}(1,1,0,0,\mathscr{T}) \geqslant F - (E+2) + 1 = V^+. \tag{3.17}$$

Combining Equations (3.16) and (3.17), it follows that the dimension theorem is proved, i.e.,

$$\dim \overline{\mathbf{S}}(1,1,0,0,\mathscr{T}) = V^+.$$

■

3.4 Basis functions

We can construct a set of basis functions $\{b_i(x,y)\}_{i=1}^{V^+}$ for the spline space $\overline{\mathbf{S}}(1,1,0,0,\mathscr{T})$. The basis functions should have the following properties:

1. **Compact Support**: For any i, $b_i(x,y)$ has a support as small as possible;

2. **Nonnegativity**: For any i, $b_i(x,y) \geqslant 0$;

3. **Partition of Unity**: If $\mathscr{T}$ is an extension of some T-mesh $\mathscr{T}_0$ with respect to the spline space $\overline{\mathbf{S}}(1,1,0,0,\mathscr{T})$, and the region occupied by $\mathscr{T}_0$ is Ω, then

$$\sum_{i=1}^{V^+} b_i(x,y) = 1, \quad (x,y) \in \Omega.$$

The basis functions with those properties can be constructed as follows: Suppose the crossing vertices in $\mathscr{T}$ are v_i with coordinate (x_i, y_i), $i = 1, \ldots, V^+$. Then we require the function $b_i(x,y)$ satisfy $b_i(x_j, y_j) = \delta_{ij}$. According to the dimension theorem 3.4 and the first part in its proof, $b_i(x,y)$ is determined uniquely. All the functions $b_i(x,y)$ form a set of basis functions of the spline space, which have the former three properties. In fact, it is easy to show that Properties 1 and 2 are satisfied. Now we prove Property 3.

Theorem 3.5 *Given a regular T-mesh $\mathscr{T}$, which occupies a rectangle Ω, its extension associated with $\mathbf{S}(1,1,0,0,\mathscr{T})$ is $\mathscr{T}^\varepsilon$. The crossing vertices in $\mathscr{T}^\varepsilon$ are v_i with coordinate (x_i, y_i), $i = 1, \ldots, V^+$. The function set $\{b_i(x,y)\}_{i=1}^{V^+} \subset \overline{\mathbf{S}}(1,1,0,0,\mathscr{T}^\varepsilon)$ satisfy $b_i(x_j, y_j) = \delta_{ij}$. Then*

$$\sum_{i=1}^{V^+} b_i(x,y) = 1, \quad (x,y) \in \Omega. \tag{3.18}$$

Proof: Let

$$f(x,y) = \sum_{i=1}^{V^+} b_i(x,y),$$

and ℓ denote the boundary of $\mathscr{T}$. In order to show Equation (3.18) holds, we first prove $f|_\ell \equiv 1$. Because the vertices on ℓ are crossing vertices in $\mathscr{T}^\varepsilon$, it follows that f reaches 1 on these vertices. Since $f|_\ell$ is a piecewise linear function with knots being these vertices, it follows that $f|_\ell \equiv 1$. Then, in the following, we prove that, for any $(x,y) \in \Omega$, $f(x,y) = 1$. Consider the function

$$g(x,y) = \begin{cases} f(x,y) - 1 & (x,y) \in \Omega \\ 0 & \text{otherwise} \end{cases}$$

Since $f|_\ell \equiv 1$, one has $g \in \overline{\mathbf{S}}(1,1,0,0,\mathscr{T})$. But g is zero at all the crossing vertices of $\mathscr{T}$, it follows that, according to the proof of the dimension theorem 3.4, we have $g \equiv 0$, i.e., $f(x,y) = 1$, $(x,y) \in \Omega$. ∎

Remarks: There are some interesting problems open here.

1. How to directly specify the former basis functions in every cell of a general T-mesh?
2. How to evaluate the function or the surface which is represented in the linear combination of the former basis functions?
3. What is the “knot” insertion algorithm in this spline space?

Though the space is just bilinear, the solutions to these problems will possibly hint us how to do in higher degree spline spaces over T-meshes.

4 A Lower Bound of the Dimension of $\overline{\mathbf{S}}(2,2,1,1,\mathscr{T})$

We can apply a similar method proposed in the proof of the dimension theorem of $\overline{\mathbf{S}}(1,1,0,0,\mathscr{T})$ in the former section to the dimension analysis of $\overline{\mathbf{S}}(2,2,1,1,\mathscr{T})$. After that, we can obtain a lower bound of the dimension, i.e.,

$$\dim \overline{\mathbf{S}}(2,2,1,1,\mathscr{T}) \geqslant V^+ - E + 1. \tag{4.1}$$

4.1 Some lemmas

Now we consider the operator of mixed partial derivative as follows:

$$\mathcal{D} := \frac{\partial^2}{\partial x \partial y} : \overline{\mathbf{S}}(2,2,1,1,\mathscr{T}) \to \overline{\mathbf{S}}(1,1,0,0,\mathscr{T}).$$

Here $\mathcal{D}$ is injective as well. The operator $\mathcal{I}(g)$ is defined in the same way as Equation (3.5). The following lemmas discuss the constraints ensuring $\mathcal{I}(g) \in \mathbf{S}(2,2,1,1,\mathscr{T})$ for any $g \in \mathbf{S}(1,1,0,0,\mathscr{T})$.

Lemma 4.1 *Given a regular T-mesh $\mathscr{T}$, let the coordinate of the end-points of any horizontal l-edges l_i^h be (x_{i1}^h, y_i^h) and (x_{i2}^h, y_i^h), $i = 0, 1, \ldots, m$, and the coordinate of the end-points of any vertical l-edge l_j^v be (x_j^v, y_{j1}^v) and (x_j^v, y_{j2}^v), $j = 0, 1, \ldots, n$. For any $g \in \overline{\mathbf{S}}(1,1,0,0,\mathscr{T})$, it follows that*

$$\mathcal{I}(g) \in \overline{\mathbf{S}}(2,2,1,1,\mathscr{T}) \Leftrightarrow \begin{cases} \displaystyle\int_{x_{i1}^h}^{x_{i2}^h} \frac{\partial}{\partial y} g(s, y_i^h-) ds = \int_{x_{i1}^h}^{x_{i2}^h} \frac{\partial}{\partial y} g(s, y_i^h+) ds, \quad i = 0, 1, \ldots, m, \text{ and} \\ \displaystyle\int_{y_{j1}^v}^{y_{j2}^v} \frac{\partial}{\partial x} g(x_j^v-, t) dt = \int_{y_{j1}^v}^{y_{j2}^v} \frac{\partial}{\partial x} g(x_i^v+, t) dt, \quad j = 0, 1, \ldots, n. \end{cases}$$

Proof: We first prove the necessity $\Longrightarrow$. Let $f(x, y) = \mathcal{I}(g)(x, y)$. Without loss of generality, we only

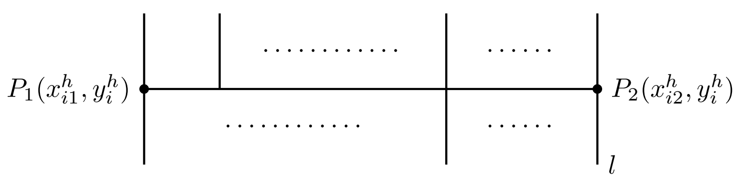


Figure 5: A horizontal l-edge (2)

prove that, when $f \in \overline{\mathbf{S}}(2,2,1,1,\mathscr{T})$, the constraints corresponding to horizontal l-edges are satisfied. As shown in Figure 5, the two end-points of the horizontal l-edge are $P_1(x_{i1}^h, y_i^h)$ and $P_2(x_{i2}^h, y_i^h)$. The vertical edge on which P_2 lies is l. Because $f \in \overline{\mathbf{S}}(2,2,1,1,\mathscr{T})$, it follows that $f(x_{i2}^h, y)$ is a quadratic polynomial with respect to the variable y in a neighborhood of P_2 along l. Hence,

$$\begin{aligned} f(x_{i2}^h, y_i^h-) &= f(x_{i2}^h, y_i^h+), \\ \frac{\partial}{\partial y} f(x_{i2}^h, y_i^h-) &= \frac{\partial}{\partial y} f(x_{i2}^h, y_i^h+), \\ \frac{\partial^2}{\partial y^2} f(x_{i2}^h, y_i^h-) &= \frac{\partial^2}{\partial y^2} f(x_{i2}^h, y_i^h+). \end{aligned}$$

According to the definition of f and the continuous of g, the first two equations hold trivially. Substituting the definition of f into the last equation, one has

$$\int_{-\infty}^{x_{i2}^h} \frac{\partial}{\partial y} g(s, y_i^h-) ds = \int_{-\infty}^{x_{i2}^h} \frac{\partial}{\partial y} g(s, y_i^h+) ds. \tag{4.2}$$

Extend the current l-edge to the left boundary of the T-mesh. If there does not exist any other l-edges on the extension, then in every cell which intersects the extension on the left side of (x_{i1}^h, y_i^h), g is a single bilinear function. Hence it follows that

$$\int_{-\infty}^{x_{i1}^h} \frac{\partial}{\partial y} g(s, y_i^h-) ds = \int_{-\infty}^{x_{i1}^h} \frac{\partial}{\partial y} g(s, y_i^h+) ds. \tag{4.3}$$

Subtracting Equation (4.3) from Equation (4.2), one gets

$$\int_{x_{i1}^h}^{x_{i2}^h} \frac{\partial}{\partial y} g(s, y_i^h-)ds = \int_{x_{i1}^h}^{x_{i2}^h} \frac{\partial}{\partial y} g(s, y_i^h+)ds.$$

This proves that the equation in the lemma corresponding to the current l-edge holds. If there exists some other l-edges on the extension, then we consider these l-edges one by one from left to right. We can prove that the former equations hold for these l-edges. Hence for the current l-edge, according to Equation (4.2), the same equation holds as well.

Now we prove the sufficiency $\Longleftarrow$. We will show that, for any $g \in \overline{\mathbf{S}}(1, 1, 0, 0, \mathscr{T})$, if the following equations hold:

$$\int_{x_{i1}^h}^{x_{i2}^h} \frac{\partial}{\partial y} g(s, y_i^h-)ds = \int_{x_{i1}^h}^{x_{i2}^h} \frac{\partial}{\partial y} g(s, y_i^h+)ds, \quad i = 0, 1, \ldots, m, \tag{4.4}$$

$$\int_{y_{j1}^v}^{y_{j2}^v} \frac{\partial}{\partial x} g(x_j^v-, t)dt = \int_{y_{j1}^v}^{y_{j2}^v} \frac{\partial}{\partial x} g(x_i^v+, t)dt, \quad j = 0, 1, \ldots, n, \tag{4.5}$$

then f is a single biquadratic polynomial in every cell of $\mathscr{T}$.

Otherwise, suppose in some cell c_i, f is piecewise with horizontal discontinuous-lines $y = y_{i_1}, \ldots, y = y_{i_{n_i}}$ and vertical discontinuous-lines $x = x_{i_1}, \ldots, x = x_{i_{m_i}}$ of the partial derivatives of order two. Let $\mathscr{C}_y = \bigcup_i \{y_{i_1}, \ldots, y_{i_{n_i}}\}$, $\mathscr{C}_x = \bigcup_i \{x_{i_1}, \ldots, x_{i_{m_i}}\}$. Then according to the assumption, $\mathscr{C}_x \bigcup \mathscr{C}_y$ is non-empty. Without loss of generality, we assume $\mathscr{C}_y$ is non-empty. Let $\bar{y} = \min\limits_{y \in \mathscr{C}_y} y$. Suppose c_i is the leftmost cell which takes $y = \bar{y}$ as its inner discontinuous-line of the partial derivatives of order two with respect to x. Denote the x coordinate of the left boundary edge of c_i is x_0, and $(x_1, \bar{y})$ is a point on the discontinuous-line, where $x_1 > x_0$. Then it follows that

$$\int_{-\infty}^{x_0} \frac{\partial}{\partial y} g(s, \bar{y}-)ds = \int_{-\infty}^{x_0} \frac{\partial}{\partial y} g(s, \bar{y}+)ds.$$

Since $(x_1, \bar{y})$ is a discontinuous point, it follows that

$$\int_{-\infty}^{x_1} \frac{\partial}{\partial y} g(s, \bar{y}-)ds \neq \int_{-\infty}^{x_1} \frac{\partial}{\partial y} g(s, \bar{y}+)ds.$$

Hence

$$\int_{x_0}^{x_1} \frac{\partial}{\partial y} g(s, \bar{y}-)ds \neq \int_{x_0}^{x_1} \frac{\partial}{\partial y} g(s, \bar{y}+)ds.$$

This contradicts with the fact that $\frac{\partial}{\partial y} g(x, y)$ is continuous inside the cell c_i. Therefore we have proved that f is a single biquadratic polynomial in every cell of $\mathscr{T}$.

In order to prove $f \in \overline{\mathbf{S}}(2, 2, 1, 1, \mathscr{T})$, one needs further to verify that f satisfies HBC. In fact, with the same approach, one can prove that f is a single biquadratic polynomial outside $\mathscr{T}$. On the other hand, $f(x, y)$ is zero when $x \leqslant x_0, y \leqslant y_0$, where (x_0, y_0) is the coordinate of the left-bottom corner of the T-mesh $\mathscr{T}$. Hence f is everywhere zero outside $\mathscr{T}$, which ensures that f satisfies HBC. Therefore $f \in \overline{\mathbf{S}}(2, 2, 1, 1, \mathscr{T})$. ∎

According to the former lemma and the fact $\dim \overline{\mathbf{S}}(1, 1, 0, 0, \mathscr{T}) = V^+$, in order to ensure $f \in \overline{\mathbf{S}}(2, 2, 1, 1, \mathscr{T})$, there are $E+4$ constraints with V^+ under-determining coefficients, where E is the number of interior l-edges. But these constraints are not linear independent, as stated in the following lemma.

Lemma 4.2 *Given a regular T-mesh $\mathscr{T}$, whose occupied rectangle is $(x_l, x_r) \times (y_b, y_t)$. Assume the different y coordinates from all the horizontal l-edges are $y_0 < y_1 < \ldots < y_n$. For $g \in \overline{\mathbf{S}}(1, 1, 0, 0, \mathscr{T})$, it follows that*

$$\int_{x_l}^{x_r} \frac{\partial}{\partial y} g(s, y_i-)ds = \int_{x_l}^{x_r} \frac{\partial}{\partial y} g(s, y_i+)ds, \quad i = 0, \ldots, n \tag{4.6}$$

is equivalent with

$$\int_{x_l}^{x_r} g(s, y_i)ds = 0, \quad i = 1, \ldots, n-1. \tag{4.7}$$

The similar conclusion can be made for vertical l-edges.

Proof: We first prove the necessity. Getting started from the bottom boundary l-edge l_0 we have

$$\int_{x_l}^{x_r} \frac{\partial}{\partial y} g(s, y_0-)ds = 0, \quad \int_{x_l}^{x_r} g(s, y_0)ds = 0.$$

Then according to Equation (4.6) as $i = 0$, one gets

$$\int_{x_l}^{x_r} \frac{\partial}{\partial y} g(s, y_0+)ds = 0.$$

On the other hand, according to the piecewise bilinear definition of g, one has

$$\begin{aligned} \int_{x_l}^{x_r} \frac{\partial}{\partial y} g(s, y_0+)ds &= \int_{x_l}^{x_r} \frac{\partial}{\partial y} \left(\frac{y_1 - y}{y_1 - y_0} g(s, y_0) + \frac{y - y_0}{y_1 - y_0} g(s, y_1) \right) ds \\ &= \frac{1}{y_1 - y_0} \left(\int_{x_l}^{x_r} g(s, y_1)ds - \int_{x_l}^{x_r} g(s, y_0)ds \right) \\ &= \frac{1}{y_1 - y_0} \int_{x_l}^{x_r} g(s, y_1)ds. \end{aligned} \tag{4.8}$$

Therefore

$$\int_{x_l}^{x_r} g(s, y_1)ds = 0.$$

Recursively, one can prove that Equation (4.7) holds for every horizontal l-edge.

In order to prove the sufficiency, we take the similar deduction in Equation (4.8). One obtains

$$\begin{aligned} \int_{x_l}^{x_r} \frac{\partial}{\partial y} g(s, y_i-)ds &= \frac{1}{y_i - y_{i-1}} \left(\int_{x_l}^{x_r} g(s, y_i)ds - \int_{x_l}^{x_r} g(s, y_{i-1})ds \right), \\ \int_{x_l}^{x_r} \frac{\partial}{\partial y} g(s, y_i+)ds &= \frac{1}{y_{i+1} - y_i} \left(\int_{x_l}^{x_r} g(s, y_{i+1})ds - \int_{x_l}^{x_r} g(s, y_i)ds \right). \end{aligned}$$

Therefore Equation (4.7) ensures Equation (4.6) holds. ■

Remark: In fact, from the proof of the lemma, we can also conclude that, for horizontal l-edges,

$$\int_{x_l}^{x_r} \frac{\partial}{\partial y} g(s, y_i-)ds = \int_{x_l}^{x_r} \frac{\partial}{\partial y} g(s, y_i+)ds, \quad i = 0, \ldots, n$$

is equivalent with

$$\int_{x_l}^{x_r} \frac{\partial}{\partial y} g(s, y_i-)ds = 0, \quad i = 1, \ldots, n-1.$$

Lemma 4.2 states that there are at least two redundant constraints among those corresponding to horizontal l-edges and vertical l-edges, respectively. The following lemma tells us that we can take the constraint along four boundary l-edges as redundant ones.

Lemma 4.3 *Given a regular T-mesh $\mathscr{T}$, the rectangle occupied by it is $(x_l, x_r) \times (y_b, y_t)$, where the different y coordinates from horizontal l-edges are $y_0 < y_1 < \ldots < y_n$. Let $g \in \overline{\mathbf{S}}(1, 1, 0, 0, \mathscr{T})$. Then Equation* (4.6) *is equivalent with*

$$\int_{x_l}^{x_r} \frac{\partial}{\partial y} g(s, y_i-)ds = \int_{x_l}^{x_r} \frac{\partial}{\partial y} g(s, y_i+)ds, \quad i = 1, \ldots, n-1. \tag{4.9}$$

The similar conclusion can be made for vertical l-edges.

Proof: The necessity is obvious. In order to prove the sufficiency, we show that Equation (4.9) implies that Equation (4.7) holds. In fact, suppose

$$I_i = \int_{x_l}^{x_r} g(s, y_i) ds, i = 0, 1, \ldots, n.$$

Then $I_0 = I_n = 0$. Our object is to prove that, for any $i = 1, 2, \ldots, n-1$, $I_i = 0$. Taking the similar deduction in Equation (4.8), one gets

$$\int_{x_l}^{x_r} g(s, y_i) ds = \frac{y_{i+1} - y_i}{y_{i+1} - y_{i-1}} \int_{x_l}^{x_r} g(s, y_{i-1}) + \frac{y_i - y_{i-1}}{y_{i+1} - y_{i-1}} \int_{x_l}^{x_r} g(s, y_{i+1}) ds,$$

i.e.,

$$I_i = \frac{y_{i+1} - y_i}{y_{i+1} - y_{i-1}} I_{i-1} + \frac{y_i - y_{i-1}}{y_{i+1} - y_{i-1}} I_{i+1}$$

holds for any $i = 1, 2, \ldots, n-1$. Hence the point set $\{(y_i, I_i)\}_{i=0}^n$ is collinear, which means that there exists a linear function $f(y)$ such that $f(y_i) = I_i$, $i = 0, \ldots, n$. Since $y_0 = y_n = 0$, it follows that $f(y) \equiv 0$, which means that $I_i = 0$, $i = 1, 2, \ldots, n$. Then the sufficiency is proved. ■

4.2 A lower bound of dimensions

For a given regular T-mesh $\mathscr{T}$, which has E interior l-edges, we assume its associated tensor-product mesh $\mathscr{T}^c$ has E' interior l-edges, where $E' \leqslant E$. Suppose the horizontal l-edges $l_1, \ldots, l_k$ in $\mathscr{T}$ lie on the horizontal l-edge l in $\mathscr{T}^c$, where the vertical coordinate of l is $\bar{y}$. Then one gets

$$\int_{l_i} \frac{\partial}{\partial y} g(s, \bar{y}-) ds = \int_{l_i} \frac{\partial}{\partial y} g(s, \bar{y}+) ds, \quad i = 1, \ldots, k$$

is equivalent with

$$\int_{l_i} \frac{\partial}{\partial y} g(s, \bar{y}-) ds = \int_{l_i} \frac{\partial}{\partial y} g(s, \bar{y}+) ds, \quad i = 1, \ldots, k-1, \quad \int_{l} \frac{\partial}{\partial y} g(s, \bar{y}-) ds = \int_{l} \frac{\partial}{\partial y} g(s, \bar{y}+) ds.$$

According to Lemma 4.2, the constraints, along all the interior horizontal l-edges and top/bottom boundary l-edges in $\mathscr{T}^c$,

$$\int_{l} \frac{\partial}{\partial y} g(s, \bar{y}-) ds = \int_{l} \frac{\partial}{\partial y} g(s, \bar{y}+) ds$$

is equivalent with the integrations of g along all the interior horizontal l-edges in $\mathscr{T}^c$ are zero:

$$\int_{l} g(s, \bar{y}) ds = 0.$$

The similar conclusion can be made for vertical l-edges. Therefore the number of the sufficient and necessary constraints that ensure $\mathcal{I}(g) \in \overline{\mathbf{S}}(2, 2, 1, 1, \mathscr{T})$ for $g \in \overline{\mathbf{S}}(1, 1, 0, 0, \mathscr{T})$ is just E.

Moreover, in the tensor-product mesh $\mathscr{T}^c$, if the horizontal knots are $x_0 < x_1 < \cdots < x_m$, and the vertical knots are $y_0 < y_1 < \cdots < y_n$, then we have

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} g(s, t) ds dt = \sum_{i=1}^{m-1} (x_{i+1} - x_{i-1}) E_i = \sum_{j=1}^{n-1} (y_{j+1} - y_{j-1}) F_j, \tag{4.10}$$

where

$$E_i = \int_{y_0}^{y_n} g(x_i, y) dy, \quad F_j = \int_{x_0}^{x_m} g(x, y_j) dx.$$

Hence, among the former E constraints, the element number in any maximal linearly independent subset is at most $E - 1$ when these constraints are not empty (i.e., $V^+ > 0$).

Based on these analysis, a lower bound of dimensions of spline spaces $\mathbf{S}(2, 2, 1, 1, \mathscr{T})$ can be proposed as follows:

Theorem 4.4 *Given a regular T-mesh $\mathscr{T}$ with $V^+ > 0$ crossing vertices and E interior l-edges, it follows that*

$$\dim \overline{\mathbf{S}}(2, 2, 1, 1, \mathscr{T}) \geqslant V^+ - E + 1.$$

Proof: Since $V^+ > 0$, the constraints are not empty, and $E > 1$. According the former analysis and Theorem 3.4, the dimension of $\overline{\mathbf{S}}(1, 1, 0, 0, \mathscr{T})$ is V^+. For any $g \in \overline{\mathbf{S}}(1, 1, 0, 0, \mathscr{T})$, the constraints ensuring $\mathcal{I}(g) \in \overline{\mathbf{S}}(2, 2, 1, 1, \mathscr{T})$ have maximal linearly dependent subsets with element number at most $E-1$. Here both $\mathcal{D}$ and $\mathcal{I}$ are linearly injective. Therefore $\dim \overline{\mathbf{S}}(2, 2, 1, 1, \mathscr{T}) \geqslant V^+ - E + 1$. ■

The lower bound in the theorem can be reached for spline spaces over some T-meshes. For example, consider the T-mesh $\mathscr{T}$ as shown in Figure 2. The B-net method in [3] can show that $\dim \overline{\mathbf{S}}(2, 2, 1, 1, \mathscr{T}) = 1$. On the other hand, in $\mathscr{T}$, $V^+ = 6$, $E = 6$. Then $V^+ - E + 1 = 1$, which means that the lower bound is reached in this T-mesh.

Furthermore, it is easy to verify that, for tensor-product meshes, the former lower bound is exactly the same as dimensions of biquadratic spline spaces over the meshes, if $V^+ - E + 1 \geqslant 0$. In the next section, we will prove that, for hierarchical T-meshes, the lower bound can be reached in some cases as well.

5 Dimensions of Spline Spaces $\overline{\mathrm{S}}(2, 2, 1, 1, \mathscr{T})$ over Hierarchical T-meshes

In this section, a careful analysis on the constraints in Section 4 will help us to build a dimension formula of the spline space $\overline{\mathbf{S}}(2, 2, 1, 1, \mathscr{T})$ over a hierarchical T-mesh. Here the key procedure consists of the following components:

1. A general hierarchical T-mesh is divided into some so-called crossing-vertex-connected branches (Definition 5.1). Then the spline space over the hierarchical T-mesh can be divided into the direct sum of some subspaces, each of which is defined over a crossing-vertex-connected hierarchical branch, which is a T-mesh as well. See Subsection 5.6.

2. In a crossing-vertex-connected hierarchical T-mesh, the constraint set is proved to with defective rank exact one by the following processing:

   (a) The constraints are converted into a new form to reflect the level structure of the hierarchical T-mesh. See Subsection 5.2.

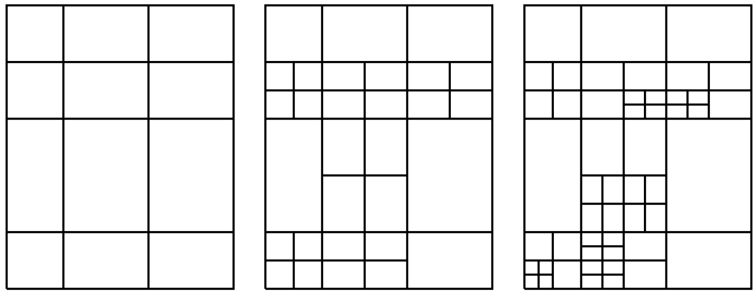
   (b) A new set of basis functions of $\overline{\mathbf{S}}(1, 1, 0, 0, \mathscr{T})$ is defined according the structure of the T-mesh, such that the occurrence of the basis function coefficients in the constraints is regularized. See Subsection 5.3 and Proposition 5.9.

   (c) All the l-edges and the corresponding constraints are ordered according to the structure of the T-mesh. For each of the l-edges or the constraints, a characteristic vertex is introduced. Write all the constraints in a vector form $(C_0, C_1, \ldots, C_T)^T$ in the increasing order. The constraint C_0 can be removed since we have known all the constraints are with defective reank at least one. Hence we need to shown that $(C_1, \ldots, C_T)^T$ is with full rank. See Subsection 5.1.3.

   (d) Assume the characteristic vertex of C_i is V_i, $i = 1, \ldots, T$. Arrange the coefficients into a vector $(\beta_1, \ldots, \beta_T, \beta_{T+1}, \ldots, \beta_M)^T$ such that β_i is the coefficient of the basis function, corresponding to V_i, $i = 1, 2, \ldots, T$, in $\overline{\mathbf{S}}(1, 1, 0, 0, \mathscr{T})$.

   (e) Then there exists a matrix $\mathbf{M} = (m_{ij})_{T\times M}$ such that $(C_1, \ldots, C_T)^T = \mathbf{M}(\beta_1, \ldots, \beta_M)^T$. It can be shown that $m_{ij} = 0$, $i > j$, and $m_{ii} \neq 0$ for any i and j.

   Hence the set of all the constraints is with defective rank one.



a. Level 0 b. Level 1 c. Level 2

Figure 6: A hierarchical T-mesh

5.1 Hierarchical T-meshes

A hierarchical T-mesh [4] is a special type of T-mesh which has a natural level structure. It is defined in a recursive fashion. One generally starts from a tensor-product mesh (level 0). From level k to level $k+1$, one subdivide a cell at level k into four subcells which are cells at level $k+1$. For simplicity, we subdivide each cell by connecting the middle points of the opposite edges with two straight lines. Figure 6 illustrates the process of generating a hierarchical T-mesh. For a hierarchical T-mesh $\mathscr{T}$, in order to emphasis its level structure in some cases, we denote the T-mesh of level k to be $\mathscr{T}^k$.

For a given hierarchical T-mesh $\mathscr{T}$, we can extend it to obtain an extended T-mesh $\mathscr{T}^\varepsilon$ associated with the spline space $\mathbf{S}(2,2,1,1,\mathscr{T})$. In the following text, hierarchical T-meshes refer to both the classical hierarchical T-meshes and their extension.

Over a hierarchical T-mesh $\mathscr{T}$, the dimension of $\overline{\mathbf{S}}(2,2,1,1,\mathscr{T})$ may be greater than the lower bound in Theorem 4.4. For example, consider the hierarchical T-mesh as shown in Figure 7, where the mesh of level 0 is a tensor-product mesh with size 3×3, and in the mesh of level 1, there exists only one cell that is subdivided. In this mesh, $V^+=5$, $E=6$. The lower bound is $V^+-E+1=0$. But the dimension of the biquadratic spline space over the mesh is at least one obviously.

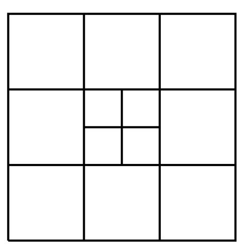


Figure 7: A hierarchical T-mesh where the dimension is greater than the lower bound

5.1.1 Crossing-vertex connected

In order to ensure the dimensions of spline spaces over hierarchical T-meshes reach the former lower bound, we need to focus on a special type of hierarchical T-meshes.

Definition 5.1 *For a regular T-mesh, if, between any two different crossing vertices, there exists a continuous poly-line, which consists of edges in the mesh, such that every joint between two neighboring horizontal*

and vertical edges on the poly-line is a crossing vertex in the mesh, then the T-mesh is called ***crossing-vertex connected****. Such poly-lines are called* ***paths*** *between the crossing-vertices.*

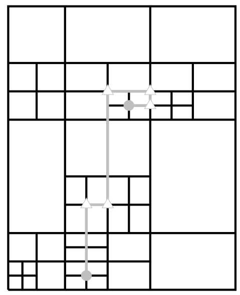


Figure 8: A hierarchical T-mesh with a path between two connected crossing vertices

For example, two crossing vertices, labeled with light-gray dots, are selected in the T-mesh as shown in Figure 8. A path between these two vertices is illustrated in light-gray as well, where the joints are shown with light-gray triangles.

Definition 5.2 *From level k to level $k+1$ as forming a hierarchical T-mesh, if there exists a cell of level k to be subdivided, but all its horizontal and vertical neighboring cells of level k remain unchanged, then the cell is called an* ***isolated subdivided cell****.*

For a hierarchical T-mesh, it is crossing-vertex connected if and only if, in any level of forming the hierarchical T-mesh, there do not exist any isolated subdivided cells.

In Subsections 5.2–5.5, we will prove that the lower bound of dimension proposed in Theorem 4.4 is exactly the dimension of $\overline{\mathbf{S}}(2, 2, 1, 1, \mathscr{T})$ over a crossing-vertex connected hierarchical T-mesh. Then in Subsection 5.6, a dimension theorem can be proposed to calculate the dimension of the spline space over a general hierarchical T-mesh by dividing it into the union of some crossing-vertex connected hierarchical branches.

5.1.2 Level numbers of edges, l-edges, and crossing-vertices

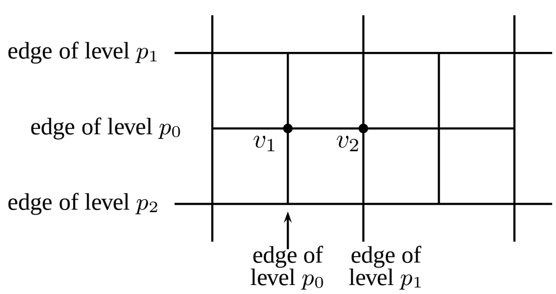


Figure 9: Level numbers of edges, l-edges, and crossing vertices

We assign every edge or l-edge in a hierarchical T-mesh with a level number as same as the level number of the T-mesh where the edge or l-edge just appears. An l-edge of level k consists of edges of level k. The extension of an edge in the extended T-mesh is assigned with the same level as its source. The new adding l-edges in the extended T-mesh are assigned with level 0.

A crossing vertex is assigned with two level numbers, denoted as (k_h, k_v), corresponding to the level number k_h of the horizontal l-edge and the level number k_v of the vertical l-edge where the vertex lies, respectively. k_h and k_v are called horizontal level and vertical level, respectively, of the crossing vertex.

For example, in the hierarchical T-mesh $\mathscr{T}$ as shown in Figure 9, suppose the middle horizontal l-edge is with level p_0, and its upper and lower neighboring l-edges are with levels p_1 and p_2, respectively. Here it should be $p_0 > p_1$, $p_0 > p_2$. Because v_1 is the intersection of two l-edges with level p_0, its level is (p_0, p_0). v_2 is the intersection of a horizontal l-edge of level p_0 and a vertical l-edge of level p_1. Hence its level is (p_0, p_1).

5.1.3 An ordering on interior l-edges

In order to sort the constraints reasonably, we introduce a partial ordering on interior l-edges in a hierarchical T-mesh. This order will be used in the proof of Theorem 5.10 to facilitate the rank analysis of the constraints. Before that, we propose the following definition.

Definition 5.3 *In a hierarchical T-mesh $\mathscr{T}$, two interior l-edges are* ***continuous*** *if they intersect in a crossing vertex of $\mathscr{T}$. An l-edge set S is* ***connected*** *if, for any two l-edges ℓ_0 and ℓ_1 in S, there exists a* ***continuous series of l-edges*** *$l_0, l_1, \ldots, l_k$ in S between ℓ_0 and ℓ_1, i.e., $l_0 = \ell_0$, $l_k = \ell_1$, and l_i and l_{i+1} are continuous for any $i = 0, 1, \ldots, k-1$.*

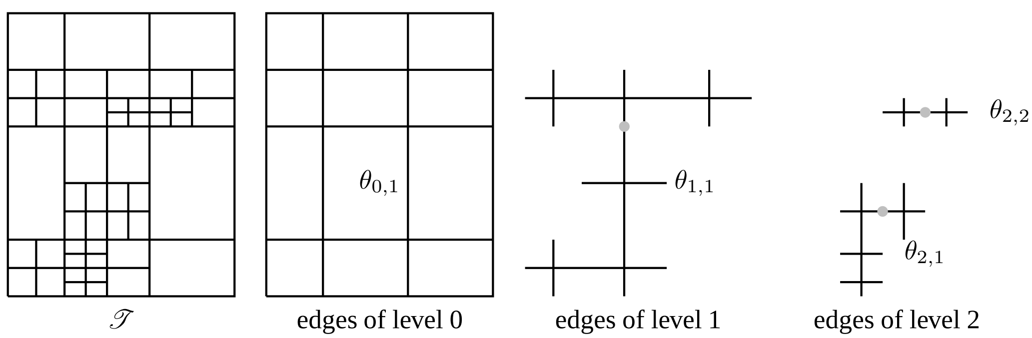


Figure 10: A hierarchical T-mesh with its decomposition according to the edge levels

Consider a crossing-vertex connected hierarchical T-mesh $\mathscr{T}$. Fix a level number $k \geqslant 0$. Then all the l-edges with level k is possibly not connected. See Figure 10 for an example, where all the edges of level 2 are not connected. We assume that they form some maximal connected branches. Denote these branches to be $\theta_{k,i}$, $i = 1, \ldots, T_k$. Hence in the example of Figure 10, we have $T_0 = 1$, $T_1 = 1$, and $T_2 = 2$. Because $\mathscr{T}$ is crossing-vertex connected, when $k > 1$, there exists at least one crossing vertex with level number (k, j) or (j, k) on some l-edge in the branch $\theta_{k,i}$, where $j < k$. See Figure 10 for examples, where the specified crossing vertex is shown in light-gray. This crossing vertex is the intersection between two l-edges of level k and j, where the l-edge of level j is called an **entering l-edge** of the branch, which is connected with every l-edge in $\theta_{k,i}$. The entering l-edge of all the l-edges of level zero is defined to be any l-edge of level zero.

Fixed an entering l-edge ℓ_0 of $\theta_{k,i}$. For an l-edge ℓ in $\theta_{k,i}$, there exists many continuous series of l-edges in $\theta_{k,i}$ connecting ℓ and ℓ_0. The number of the l-edges in a series is called the length of the series, and the minimal length of all the series connecting ℓ and ℓ_0 is called the **distance between ℓ and ℓ_0**, denoted as $\operatorname{dist}(\ell, \ell_0)$. Suppose $e_0 = \ell_0, e_1, \ldots, e_s = \ell$ is an l-edge series with the minimal length among the continuous series connecting ℓ and ℓ_0. Then the intersection point between $e_s = \ell$ and e_{s-1} is a crossing vertex on ℓ, which is called a **characteristic vertex of the l-edge ℓ**. Its level is (k, j) or (j, k), where $j \leqslant k$.

After having selected the entering l-edges for all the connected branches $\theta_{k,i}$, we introduce a partial ordering $<_1$ on all the interior l-edges in $\mathscr{T}$. For any two interior l-edges ℓ_1 and ℓ_2 with levels k_1 and k_2, respectively, where ℓ_j is in the branch θ_{k_j,i_j}, $j = 1, 2$, we define $\ell_1 <_1 \ell_2$ if

1. $k_1 < k_2$, or
2. $k_1 = k_2$ and $i_1 < i_2$, or
3. $k_1 = k_2$ and $i_1 = i_2$, $\operatorname{dist}(\ell_1, \ell_0) < \operatorname{dist}(\ell_2, \ell_0)$, where ℓ_0 is the entering l-edge of the connected branch in which ℓ_1 and ℓ_2 lies, since these two l-edges are in the same connected branch.

This order is not total, because, in Case 3, it is possible that $\operatorname{dist}(\ell_1, \ell_0) = \operatorname{dist}(\ell_2, \ell_0)$ for two different interior l-edges.

5.2 Conversion of constraints

In Section 4 we have proposed some different versions of the necessary and sufficient conditions that ensure $\mathcal{I}(g) \in \overline{\mathbf{S}}(2,2,1,1,\mathscr{T})$ for any $g \in \overline{\mathbf{S}}(1,1,0,0,\mathscr{T})$. In order to facilitate the latter analysis, we take the following notation and the constraints.

For any regular T-mesh $\mathscr{T}$, denote its occupied rectangle to be $(x_l, x_r) \times (y_b, y_t)$. $\mathscr{T}^c$ is an associated tensor-product mesh with $\mathscr{T}$. Assume the y coordinates of all the horizontal l-edges in $\mathscr{T}^c$ to be $y_b = y_0 < y_1 < \cdots < y_m < y_{m+1} = y_t$, and the x coordinates of all the vertical l-edges in $\mathscr{T}^c$ to be $x_l = x_0 < x_1 < \cdots < x_n < x_{n+1} = x_r$. For $i = 0, 1, \ldots, m+1$, assume the horizontal l-edges $l^h_{i1}, \ldots, l^h_{i\alpha_i}$ are with y coordinate $y = y_i$. Here $l^h_{i1}, \ldots, l^h_{i\alpha_i}$ are sorted from left to right. On the other hand, for $j = 0, 1, \ldots, n+1$, assume the vertical l-edges $l^v_{j1}, \ldots, l^v_{j\beta_j}$ are with x coordinate $x = x_j$. Here $l^v_{j1}, \ldots, l^v_{j\beta_j}$ are sorted from bottom to top. It follows that $\alpha_0 = \alpha_{m+1} = \beta_0 = \beta_{n+1} = 1$, $\alpha_1 + \cdots + \alpha_m + \beta_1 + \cdots + \beta_n = E$.

With these notation, according to Lemma 4.3, we can allocate the E constraints to the interior l-edges in $\mathscr{T}^c$ in the following fashion: To any interior horizontal l-edge $y = y_i$, the corresponding constraints are

$$\int_{l^h_{ik}} \frac{\partial}{\partial y} g(s, y_i-) ds = \int_{l^h_{ik}} \frac{\partial}{\partial y} g(s, y_i+) ds, \quad k = 1, \ldots, \alpha_i. \tag{5.1}$$

To any interior vertical l-edge $x = x_j$, the corresponding constraints are

$$\int_{l^v_{jk}} \frac{\partial}{\partial x} g(x_j-, t) dt = \int_{l^v_{jk}} \frac{\partial}{\partial x} g(x_j+, t) dt, \quad k = 1, \ldots, \beta_j. \tag{5.2}$$

Furthermore, by applying the similar deduction with Equation (4.8), Equations (5.1) and (5.2) are equivalent with

$$(y_{i+1} - y_{i-1}) \int_{x^h_{ik0}}^{x^h_{ik1}} g(s, y_i) ds = (y_{i+1} - y_i) \int_{x^h_{ik0}}^{x^h_{ik1}} g(s, y_{i-1}) ds + (y_i - y_{i-1}) \int_{x^h_{ik0}}^{x^h_{ik1}} g(s, y_{i+1}) ds, \quad k = 1, \ldots, \alpha_i, \tag{5.3}$$

$$(x_{j+1} - x_{j-1}) \int_{y^v_{jk0}}^{y^v_{jk1}} g(x_j, t) dt = (x_{j+1} - x_j) \int_{y^v_{jk0}}^{y^v_{jk1}} g(x_{j-1}, t) dt + (x_j - x_{j-1}) \int_{y^v_{jk0}}^{y^v_{jk1}} g(x_{j+1}, t) dt, \quad k = 1, \ldots, \beta_j, \tag{5.4}$$

respectively, where x^h_{ik0} and x^h_{ik1} are the x coordinates of the two end-points of l^h_{ik}, and y^v_{jk0} and y^v_{jk1} are the y coordinates of the two end-points of l^v_{jk}.

Now we focus on hierarchical T-meshes, on which the corresponding constraints can be converted into a form to reflect the level structure of the T-mesh. At first, we introduce two definitions.

In a hierarchical T-mesh $\mathscr{T}$, select any horizontal l-edge l with level $k > 0$. Hence l appears in $\mathscr{T}$ since $\mathscr{T}^k$. On l, there exists one or more crossing vertices with vertical level k. These crossing vertices are the center of some inserted crossing from $\mathscr{T}^{k-1}$ to $\mathscr{T}^k$. The l-edge l consists of the horizontal edges of these inserted crossing. It follows that the vertical edges of the inserted crossing intersect with two l-edges l^t and l^b in $\mathcal{T}^{k-1}$. Here we assume l^b with level k^b lies under l, and l^t with level k^t lies above l.

Definition 5.4 *For a horizontal l-edge l with level $k > 0$, the l-edges l^b and l^t are defined as the above description. Then l^b and l^t are called the* ***support l-edges*** *of l. For an horizontal l-edge l with level 0, its support l-edges are defined to be the two nearest horizontal l-edges with level 0, which lie above and under l, respectively. For vertical l-edges, the support l-edges are defined in a similar way.*

It is obvious that, for any horizontal/vertical l-edge l, two vertical/horizontal l-edges, which are through the two end-points of l, intersect with the two support l-edges of l, where there is not other crossing vertex between the intersection points and the end-points along the two vertical/horizontal l-edges.

Definition 5.5 *For the E constraints ensuring $\mathcal{I}(g) \in \overline{\mathbf{S}}(2,2,1,1,\mathscr{T})$, the number r of its maximal linearly independent subset is called the* ***rank of the constraints****. Here $E - r$ is called the* ***defective rank of the constraints****.*

In fact, if we specify a set of basis functions of $\overline{\mathbf{S}}(1,1,0,0,\mathscr{T})$, and undetermine the coefficients of g with these basis functions, then these constraints can be represented into the linear combinations of these coefficients. Hence the rank of the constraints is the rank of the corresponding coefficient matrix.

In Equations (5.3) and (5.4), a constraint along an l-edge is represented into the linear combination of three integrations, which are along the current l-edge and its two neighboring horizontal/vertical lines, respectively, with the same integration limits. Here the horizontal/vertical lines share the same y/x coordinates as the two-sided nearest horizontal/vertical l-edges to the current l-edge. In the following lemma, we will convert these constraint into a new form, such that every constraint along an l-edge is a linear combination of three integration, which are along the current l-edge and its support l-edges, respectively. With this form, every constraint will involve undetermined coefficients in a way easy for rank determining.

Lemma 5.6 *Given a hierarchical T-mesh $\mathscr{T}$, select an interior horizontal l-edge l_i^h. Suppose the y coordinate of l_i^h is y_i^h, the x coordinates of its two end-points are x_{i1}^h and x_{i2}^h, and the y coordinates of its support l-edges are y_i^{hb} and y_i^{ht}. Then Equation* (5.3) *holds for all the interior horizontal l-edges if and only if, for all the interior horizontal l-edges, the following equation holds:*

$$(y_i^{ht} - y_i^{hb}) \int_{x_{i1}^h}^{x_{i2}^h} g(s, y_i^h) ds = (y_i^{ht} - y_i^h) \int_{x_{i1}^h}^{x_{i2}^h} g(s, y_i^{hb}) ds + (y_i^t - y_i^{hb}) \int_{x_{i1}^h}^{x_{i2}^h} g(s, y_i^{ht}) ds. \tag{5.5}$$

For interior vertical l-edges, the conclusion is similar.

Proof: Necessity. Suppose the interior horizontal l-edge l^h is with level k, and its two support l-edges are l^b and l^t. Between l^h and l^b, there are some other l-edges. Suppose the different y coordinates these l-edges are $\bar{y}_0 < \bar{y}_1 < \cdots < \bar{y}_{\bar{n}}$, where $\bar{y}_0$ and $\bar{y}_{\bar{n}}$ correspond to l^b and l^h, respectively. Suppose again the different y coordinates of the l-edges between l^h and l^t are $\bar{y}_{\bar{n}} < \bar{y}_{\bar{n}+1} < \cdots < \bar{y}_{\bar{n}+\bar{m}}$, where $\bar{y}_{\bar{n}+\bar{m}}$ corresponds to l^t. According to the definition of the support l-edges, these horizontal l-edges excluding l^h, l^b and l^t are with level greater than k. Hence there does not exist one of them such that its vertical projection onto l^h takes one of the end-points of l^h as its interior point. Therefore, according to the constraints in Equation (5.3) corresponding to these l-edges, we can conclude that $(\bar{y}_{i-1}, I_{i-1})$, $(\bar{y}_i, I_i)$, $(\bar{y}_{i+1}, I_{i+1})$ are collinear, where $i = 1, \ldots, \bar{m} + \bar{n} - 1$, and

$$I_i = \int_{x_{i1}^h}^{x_{i2}^h} g(s, \bar{y}_i) ds.$$

Then $(\bar{y}_0, I_0)$, $(\bar{y}_{\bar{n}}, I_{\bar{n}})$, $(\bar{y}_{\bar{n}+\bar{m}}, I_{\bar{n}+\bar{m}})$ are collinear, which means the corresponding constraint in Equation (5.5) holds.

Sufficiency. We prove it in an inductive fashion from the highest level to the lowest level. We will show that, for any given level k_0, if the constraints as defined in Equation (5.5) hold corresponding to the l-edges of level $k \geqslant k_0$, then the constraints in Equation (5.3) corresponding to all the l-edges of level k_0 hold as well.

Suppose the maximal level number in the hierarchical T-mesh $\mathscr{T}$ is M. For an interior horizontal l-edge with level M, since there do not exist other l-edges between the current l-edge and its support l-edges, the corresponding constraint defined in Equation (5.3) is the same as one defined in Equation (5.5).

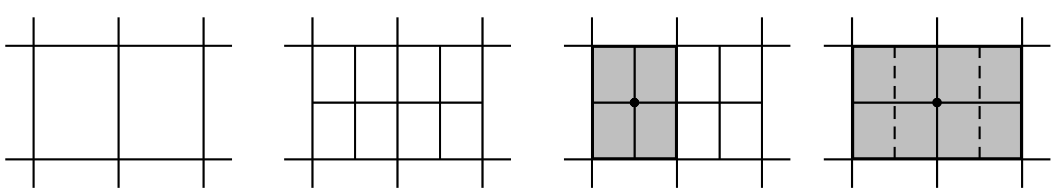
Now we assume that, for all the interior horizontal l-edges with level greater than k, if the corresponding constraints in Equation (5.5) hold, then the corresponding constraints in Equation (5.3) hold as well. Select an arbitrary interior horizontal l-edge l^h with level k, whose support l-edges are l^b and l^t. The two-sided nearest l-edges of l^h are with y coordinates h_0 and h_1. Since the level numbers of the other horizontal l-edges between l^h and l^b are greater than k, according to the inductive assumption, the integration values along these horizontal lines with the integration interval $[x_{i1}^h, x_{i2}^h]$ are collinear with respect to their y coordinates. Especially, the corresponding integration values along l^h, l^b and $y = h_0$ are collinear with respect to their y coordinates. Similarly, the integration values along l^h, l^t and $y = h_1$ are collinear with respect to their y coordinates as well. Hence the integration values along l^h, $y = h_0$ and $y = h_1$ are collinear with respect to their y coordinates. This finishes the proof of the sufficiency. ■

In the next subsection we will specify a proper set of basis functions of $\overline{\mathbf{S}}(1, 1, 0, 0, \mathscr{T})$ such that the coefficients of g under these basis functions appear in these constraints in a regular form.

5.3 Hierarchical basis functions of $\overline{\mathbf{S}}(1, 1, 0, 0, \mathscr{T})$

In Subsection 3.4 an approach has been proposed to specify a set of basis functions of $\overline{\mathbf{S}}(1, 1, 0, 0, \mathscr{T})$ with many good properties. Now we specify a new set of basis functions for $\overline{\mathbf{S}}(1, 1, 0, 0, \mathscr{T})$, which does not poss the unity-partition property. But under this set, the former constraints in Lemma 5.6 will appear in a regular form, which facilitates the rank determination of the constraints.

The new set of basis functions is defined level by level when forming the hierarchical T-mesh. Every basis function is associated with a crossing vertex. At the level 0, we consider the level 0 T-mesh $\mathscr{T}^0$, and introduce the functions to be standard bilinear tensor-product B-splines. Hence every function can be associated with a crossing vertex in $\mathscr{T}^0$ in a way that the function reaches one at the crossing vertex, and zero at all the other crossing vertices of $\mathscr{T}^0$. Use the set $\mathcal{B}^0$ to denote all these functions. Suppose the current level number is $k \geqslant 1$. Consider a new-coming crossing vertex in this level.



a) A part in $\mathscr{T}^{k-1}$ b) After inserting crosses c) Support example 1 d) Support example 2

Figure 11: The support of the hierarchical basis functions

1. If its level is (k, k), then the crossing vertex must be the center of an inserted cross into a cell c of $\mathcal{T}^{k-1}$. Hence a function can be defined associated with the crossing vertex such that the function reaches one at the vertex and its support is c. See Figure 11.c) for an example, where the filled region with light-gray is the support of the specified function associated with the new crossing vertex labeled

with $\bullet$. Here the discontinuous of the derivatives $\partial/\partial x$ and $\partial/\partial y$ appears on the edges of the inserted cross.

2. Otherwise, the crossing vertex is the middle point of an edge e in $\mathscr{T}^{k-1}$, where e is the common edge of two neighboring cells c_1 and c_2, each of which is subdivided by inserting a cross from level $k-1$ to k. Then a function can be defined such that the function reaches one at the current crossing vertex, its support is exactly $c_1 \cup c_2$, and its derivative discontinuous in the support lies only on the lines through the current crossing vertex. See Figure 11.d) for an example, where the derivatives $\partial/\partial x$ and $\partial/\partial y$ are continuous along the dashed edges.

All the functions introduced at level k are denoted to be $\mathcal{B}^k$.

Lemma 5.7 *Suppose the maximal level number of the hierarchical T-mesh $\mathscr{T}$ is M. Define $\mathcal{B} = \bigcup_{k=0}^{M} \mathcal{B}^k$. Then $\mathcal{B}$ is a set of basis functions of $\overline{\mathbf{S}}(1,1,0,0,\mathscr{T})$.*

Proof: According to the definition of the functions in $\mathcal{B}^k$, it follows that the number of the functions in $\mathcal{B}$ is exactly the number of crossing vertices in $\mathscr{T}$. Moreover, these functions are in the spline space $\overline{\mathbf{S}}(1,1,0,0,\mathscr{T})$. Therefore, in order to prove that $\mathcal{B}$ forms a set of basis functions of $\overline{\mathbf{S}}(1,1,0,0,\mathscr{T})$, one just needs to show the functions in $\mathcal{B}$ are linearly independent.

Assume the region occupied by $\mathscr{T}$ is Ω. Let $\mathcal{B}^k = \{b_1^k(x,y),\ldots,b_{n_k}^k(x,y)\}$. Suppose a set of coefficients of α_i^k ensure that

$$f(x,y) := \sum_{k=0}^{M}\sum_{i=0}^{n_k} \alpha_i^k b_i^k(x,y) = 0, \quad (x,y) \in \Omega.$$

Now we will prove that $\alpha_i^k = 0$ for any i and k.

Consider the values of f at every crossing vertex v_i^0 with level $(0,0)$. Since $f \equiv 0$, it follows that $f(v_i^0) = 0$. On the other hand, the function $b_i^0(x,y)$ in $\mathcal{B}^0$ associated with v_i^0 is one at v_i^0, and all the other functions vanish at v_i^0. Hence $f(v_0^i) = \alpha_i^0 b_i^0(v_0^i) = \alpha_i^0$, i.e., $\alpha_i^0 = 0$. Hence, we have

$$f(x,y) = \sum_{k=1}^{M}\sum_{i=0}^{n_k} \alpha_i^k b_i^k(x,y).$$

Suppose we have proved that $\alpha_i^j = 0, i = 0,\ldots,n_j, j = 0,1,\ldots,k-1$. Now we consider the functions and their coefficients introduced at level k. Each of such the functions is associated with a crossing vertex. We first consider the functions which are associated with crossing vertices whose vertical and horizontal level numbers are different. On such a crossing vertex, all the functions are zero except the associated function with the current crossing vertex. Hence the corresponding coefficient is zero. Then we consider the other functions, which are associated with crossing vertices with the same vertical and horizontal level numbers. One can conclude that their coefficients are zero as well in a similar way. Hence all the coefficients are zero, and the functions in $\mathscr{B}$ are linearly independent. Therefore they form a set of basis functions of $\overline{\mathbf{S}}(1,1,0,0,\mathscr{T})$. ■

W will call this set of basis functions to be **the hierarchical basis functions** of $\overline{\mathbf{S}}(1,1,0,0,\mathscr{T})$.

5.4 Concretion of the constraints

For a given hierarchical T-mesh $\mathscr{T}$, suppose the hierarchical basis functions of $\overline{\mathbf{S}}(1,1,0,0,\mathscr{T})$ are

$$\{b_i^k(x,y)\} = \{c_j(x,y)\}_{j=1}^{V^+},$$

where $b_i^k(x,y)$ is associated with a crossing vertex appearing in $\mathscr{T}$ since level k.

Represent g as

$$g(x,y) = \sum_{j=1}^{V^+} \alpha_j c_j(x,y). \tag{5.6}$$

Since there is a one-to-one mapping between the basis functions and the crossing vertices, the coefficient of a basis function is called also the **coefficient of the corresponding crossing vertex**.

Now we state the characteristic of the constraints after substituting the representation of g as defined in Equation (5.6) into Equation (5.5) which ensures $\mathcal{I}(g) \in \overline{\mathbf{S}}(2,2,1,1,\mathscr{T})$. At first we consider the constraints corresponding to the interior l-edges of level 0.

Proposition 5.8 *After substituting Equation* (5.6) *into Equation* (5.5)*, and then taking a proper transformation, all the constraints corresponding to the interior horizontal l-edges of level 0 are in such a form that the nonzero terms in a constraint consist of those associated with the crossing vertices on the same l-edge.*

Proof: Each of interior horizontal l-edges with level 0 must traverse the whole mesh. Its support l-edges are with level 0 as well. Over the T-mesh $\mathscr{T}^0$, we apply Lemma 4.2 and obtain that all the constraints corresponding to the interior horizontal l-edges of level 0 are equivalent with that the integration of g along each interior horizontal l-edge is zero. For every l-edge of level 0, among all the coefficients, only those of the crossing vertices on the current l-edge are nonzero. Hence the integration can be represented into a linear combination of the coefficients of the crossing vertices on the current l-edge. ■

Then we consider the constraints corresponding to the interior l-edges of level greater than 0.

Proposition 5.9 *Suppose the current interior horizontal l-edge is with level $k > 0$. After substituting Equation* (5.6) *into Equation* (5.5)*, the nonzero coefficients of the basis functions in the corresponding constraint consist of two parts as follows:*

A. the coefficients of the crossing vertices on the current l-edge;

B. the possible coefficients of the crossing vertices with horizontal level less than k and vertical level greater than k.

Proof: Select an interior horizontal l-edge l^h with level $k > 0$. Suppose its support l-edges are l^b and l^t with levels k_1 and k_2, respectively. Assume the x coordinates of the two end-points of l^h are x_0^h and x_1^h, and the vertical l-edges through the two end-points are l_l^v and l_r^v. Then the two vertical l-edges l_l^v and l_r^v intersect with l_b and l_t. Consider the crossing vertices whose associating basis functions are nonzero on l^h, $l^b|_{[x_0^h,x_1^h]}$ or $l^t|_{[x_0^h,x_1^h]}$. These crossing vertices can be classified into the following cases with respect to their level (k_0^h, k_1^h):

1. $k_0^h < k$:
   (a) $k_1^h \leqslant k$. Because, for any $x \in [x_0^h, x_1^h]$, the corresponding three points on l^h, l^t, and l^b with the horizontal coordinates x appear simultaneously in a cell of $\mathscr{T}^{k-1}$ (including its boundary), it follows that for the current basis function $b(x,y)$, Equation (5.5) holds as $g = b(x,y)$. This means that the coefficients of $b(x,y)$ does not appear in the constraint after simplification.
   (b) $k_1^h > k$. This types of coefficients can appear in the constraint.
2. $k_0^h = k$: This type of crossing vertex has no contribution to the constraints, unless it lies on l^h.
3. $k_0^h > k$: The associating basis function with this type of crossing vertex vanishes on l^h, l^b and l^t between x_0^h and x_1^h. Hence the corresponding coefficient does not appear in the constraint.

Therefore the nonzero coefficients come in Cases 1(b) and 2, which correspond to Cases B and A in the proposition description, respectively. ■

We can make a similar classification on the coefficients' appearance in the constraints corresponding to interior vertical l-edges.

After these classifications, one knows that the coefficient of a crossing vertex with level (k_1, k_2) can appear in the following three types of places:

1. The constraints corresponding to the horizontal l-edge through the vertex;
2. The constraints corresponding to the vertical l-edge through the vertex;
3. If $k_1 < k_2 - 1$ (or $k_1 - 1 > k_2$), it may appear in the constraints corresponding to horizontal (or vertical) l-edges with level less than k_2 (or k_1). If $k_1 = k_2$ or $k_2 \pm 1$, this situation does not happen.

The involved l-edges in the first two cases are called **naturally appearing l-edges of the coefficient**. The involved l-edges in the third case are called **unnaturally appearing l-edges of the coefficient**. For simplify, a constraint corresponding to a horizontal/vertical l-edge of level k is simply called the **horizontal/vertical constraint of level** k in the following. For two given interior l-edges ℓ_1 and ℓ_2, their corresponding constraints are c_1 and c_2, respectively. If $\ell_1 <_1 \ell_2$, then we define $c_1 <_1 c_2$ as well.

5.5 Dimension theorem over crossing-vertex connected hierarchical T-meshes

With these preparations, now we state and prove the dimension theorem of biquadratic spline spaces over crossing-vertex connected hierarchical T-meshes.

Theorem 5.10 *Over a crossing-vertex connected hierarchical T-mesh $\mathscr{T}$ or its extension associated with $\mathbf{S}(2,2,1,1,\mathscr{T})$, where $V^+ > 0$, it follows that*

$$\dim \overline{\mathbf{S}}(2,2,1,1,\mathscr{T}) = V^+ - E + 1.$$

Proof: Recall some facts first. For any $g \in \overline{\mathbf{S}}(1,1,0,0,\mathscr{T})$, in order to ensure that $\mathcal{I}(g) \in \overline{\mathbf{S}}(2,2,1,1,\mathscr{T})$, the constraints corresponding to every interior horizontal l-edges (defined in Equation (5.5)) and vertical l-edges should be satisfied. Here we apply the hierarchical basis functions $\mathcal{B}$ of $\overline{\mathbf{S}}(1,1,0,0,\mathscr{T})$. These constraints can be represented into linear combinations of the coefficients of g under the basis functions $\mathcal{B}$. The coefficients appear in these constraints as stated in Propositions 5.8 and 5.9.

Suppose the l-edges of level k in $\mathscr{T}$ form some connected branch $\theta_{k,i}$, $i = 1, 2, \ldots, T_k$, and the entering l-edges for all the branches $\theta_{k,i}$ have been selected, denoted as $\ell_{k,i}$. Specially, all the interior l-edges of level zero just form a connected branch, and its entering l-edge is selected to any one interior l-edge of level zero. Then we can introduce a partial ordering $<_1$ on the interior l-edges and the corresponding constraints as stated in Subsection 5.1.3.

At the beginning, all the constraints are linearly dependent because Equation (4.10) shows a linear combination of these constraints with result zero. In this linear combination, each of the coefficients of level zero is nonzero. Hence the rank of the constraints remains unchanged after deleting any one constraint of level zero. Without loss of generality, we assume that the deleted constraint is corresponding to the entering l-edge $\ell_{0,1}$ of $\theta_{0,1}$. Then we focus on the remaining constraints and we will show that they are linear independent.

Sort all the remaining constraints into a non-decreasing series $C_1, C_2, \cdots, C_T$ according to the ordering $<_1$, where $T = E - 1$, and sort the coefficients $\{\alpha_i\}$ in Equation (5.6) into a series $\beta_1, \beta_2, \ldots, \beta_{V^+}$ such that β_i is a characteristic vertex of the l-edge whose corresponding constraint is C_i, $i = 1, \ldots, T$. The rest

variables $\beta_{T+1}, \ldots, \beta_{V^+}$ are arranged randomly. Then we can write these constraints into the following matrix form:

$$(C_1, C_2, \ldots, C_T)^T = \mathbf{M}\,(\beta_1, \beta_2, \ldots, \beta_{V^+})^T\,,$$

where $\mathbf{M} = (m_{ij})$. Because the characteristic vertex of an l-edge ℓ with level k is with level (k, j) or (j, k), where $j \leqslant k$, it follows that, according to Propositions 5.8 and 5.9, the coefficient β_i does not appear in the constraints $C_{i+1}, \ldots, C_T$, where $i = 1, \ldots, T-1$. Hence $m_{ij} = 0$, $i > j$. On the other hand, the matrix $\mathbf{M}$ is with $m_{ii} \neq 0$, $i = 1, \ldots, T$. This means that the matrix $\mathbf{M}$ is with full row rank, i.e., $\operatorname{rank}\mathbf{M} = T$. Therefore the dimension of the spline space $\overline{\mathbf{S}}(2, 2, 1, 1, \mathscr{T})$ is $V^+ - T = V^+ - E + 1$. This completes the proof of the theorem. ■

5.6 General Dimension Theorem

In this subsection we consider the dimension formula of $\overline{\mathbf{S}}(2, 2, 1, 1, \mathscr{T})$, where $\mathscr{T}$ is a general hierarchical T-mesh.

At first, we discuss how to divide a general hierarchical T-mesh into the union of some crossing-vertex connected hierarchical T-meshes. Suppose the given hierarchical T-mesh is $\mathscr{T}$, and let c_i, $i = 1, \ldots, C$, be all the isolated subdivided cells with level $k > 0$ in forming $\mathscr{T}$. Then the subdivision happening in c_i to form $\mathscr{T}$ will form as well a hierarchical T-mesh, denoted as $\mathscr{U}_i$. Here $\mathscr{U}_i$ occupies the same region as the cell c_i. In the following we will apply a T-mesh operation '$\backslash$'. Suppose $\mathscr{T}_i$, $i = 0, 1, \ldots, k$, are hierarchical T-meshes. Then $\mathscr{T}_0 \backslash \{\mathscr{T}_1, \ldots, \mathscr{T}_k\}$ is a new T-mesh which consists of edges and vertices in $\mathscr{T}_0$ but not in the interior of $\mathscr{T}_i$, $i = 1, \ldots, k$. Let $\mathscr{U}_0 = \mathscr{T}$, and

$$\mathscr{V}_i = \mathscr{U}_i \backslash \{\mathscr{U}_j : \mathscr{U}_j \text{ is a submesh of } \mathscr{U}_i, j = 1, \ldots, C, j \neq i\}, i = 0, 1, \ldots, C.$$

Then it is easy to verify that $\mathscr{V}_i$ is a crossing-vertex connected hierarchical T-mesh. Here $\mathscr{V}_0$ comes from $\mathscr{T}$, and the other $\mathscr{V}_i$'s come from the isolated subdivided cells with level $k > 0$. Then it follows that $\mathscr{T}$ can be seen as the disjoint union of $\mathscr{V}_i$, $i = 0, 1, \ldots, C$.

Lemma 5.11 *Suppose the hierarchical T-mesh $\mathscr{T}$ and its disjoint union of $\mathscr{V}_i$, $i = 0, 1, \ldots, C$, are defined as before. Then it follows that*

$$\overline{\mathbf{S}}(2, 2, 1, 1, \mathscr{T}) = \bigoplus_{i=0}^{C} \overline{\mathbf{S}}(2, 2, 1, 1, \mathscr{V}_i). \tag{5.7}$$

Proof: At first, we prove that the intersection of any two different subspaces among $\overline{\mathbf{S}}(2, 2, 1, 1, \mathscr{V}_i)$, $i = 0, 1, \ldots, C$ are $\{0\}$. Therefore we can define the direct sum of these subspaces. Suppose we select the submeshes $\mathscr{V}_{i_0}$ and $\mathscr{V}_{i_1}$, where $i_0 \neq i_1$. The submeshes $\mathscr{U}_{i_0}$ and $\mathscr{U}_{i_1}$ are defined as before. If any one of $\mathscr{U}_{i_0}$ and $\mathscr{U}_{i_1}$ is not a submesh of the other one, then the regions occupied by these them are disjoint, which follows that the intersection of the subspaces over $\mathscr{V}_{i_0}$ and $\mathscr{V}_{i_1}$ is just $\{0\}$. Otherwise, we assume, without loss of generality, that $\mathscr{U}_{i_0}$ is a submesh of $\mathscr{U}_{i_1}$. Then the region occupied by $\mathscr{U}_{i_0}$ is inside a cell of $\mathscr{U}_{i_1}$. Denote the cell to be c. In $\mathscr{V}_{i_1}$, the zero function is a unique function whose function values and two partial derivatives of order one are zero along the boundary of c. Hence it follows that the intersection of the subspaces over $\mathscr{V}_{i_0}$ and $\mathscr{V}_{i_1}$ is $\{0\}$ as well.

It is obvious that

$$\overline{\mathbf{S}}(2, 2, 1, 1, \mathscr{T}) \supset \bigoplus_{i=0}^{C} \overline{\mathbf{S}}(2, 2, 1, 1, \mathscr{V}_i). \tag{5.8}$$

On the other hand, for any $f \in \overline{\mathbf{S}}(2, 2, 1, 1, \mathscr{T})$, one can construct its component in each subspace as follows. For any $j = 0, 1, \ldots, C$, we can arrange all the meshes $\{\mathscr{U}_i\}_{i=1}^{C}$, each of which takes $\mathscr{U}_j$ as a submesh, in an ascending chain as follows:

$$\mathscr{U}_j = \mathscr{U}_{i_j} \subset \mathscr{U}_{i_j - 1} \subset \cdots \subset \mathscr{U}_{i_0} = \mathscr{U}_0.$$

Since $\mathscr{V}_0$ comes from $\mathscr{U}_0 = \mathscr{T}$ by deleting the subdivisions in some isolated subdivided cells, we can define a new function $\mathbb{P}_0 f$ which meets f with order one along all the edges in $\mathscr{V}_0$. Then $f - \mathbb{P}_0 f$ vanishes out of those isolated subdivided cells of $\mathscr{U}_0$. We define recursively that $\mathbb{P}_j f$ is a function in $\overline{\mathbf{S}}(2,2,1,1,\mathscr{V}_j)$ which meets the function $f - \sum_{k=i_0}^{i_j-1} \mathbb{P}_k f$ with order one along all the edges in $\mathscr{V}_j$. Then $f - \sum_{k=i_0}^{i_j} \mathbb{P}_k f$ vanishes in $\mathscr{V}_j$ except in the isolated subdivided cells of $\mathscr{U}_j$.

According to the definition of $\mathbb{P}_k$, it follows that $f - \sum \mathbb{P}_k f$ vanishes everywhere in $\mathscr{T}$. Hence $f \in \bigoplus_{i=0}^{C} \overline{\mathbf{S}}(2,2,1,1,\mathscr{V}_i)$, which means that

$$\overline{\mathbf{S}}(2,2,1,1,\mathscr{T}) \subset \bigoplus_{i=0}^{C} \overline{\mathbf{S}}(2,2,1,1,\mathscr{V}_i). \tag{5.9}$$

Combining Equations (5.8) and (5.9) together, one gets Equation (5.7). ■

With this lemma, we have the following theorem about the dimension formula of the spline spaces over general hierarchical T-meshes.

Theorem 5.12 *Suppose $\mathscr{T}$ is a hierarchical T-meshes with $\delta - 1$ isolated subdivided cells. Then*

$$\dim \overline{\mathbf{S}}(2,2,1,1,\mathscr{T}) = V^+ - E + \delta.$$

Proof: Suppose $\mathscr{U}_j$ and $\mathscr{V}_j$, $j = 0,1,\ldots,C$ are defined as in the proof of Lemma 5.11. Then it follows that $C = \delta - 1$ and, according to Lemma 5.11,

$$\dim \overline{\mathbf{S}}(2,2,1,1,\mathscr{T}) = \sum_{i=0}^{C} \dim \overline{\mathbf{S}}(2,2,1,1,\mathscr{V}_i).$$

Assume in $\mathscr{V}_i$ there are V_i^+ crossing vertices and E_i interior l-edges. Then

$$\dim \overline{\mathbf{S}}(2,2,1,1,\mathscr{T}) = \sum_{i=0}^{C} (V_i^+ - E_i + 1).$$

Since any two different meshes among V_i do not share any common crossing vertices and interior l-edges, it follows that

$$\sum_{i=0}^{C} (V_i^+ - E_i + 1) = V^+ - E + C + 1 = V^+ - E + \delta.$$

Hence we have

$$\dim \overline{\mathbf{S}}(2,2,1,1,\mathscr{T}) = V^+ - E + \delta.$$

■

5.7 Some Notes on Construction of Basis Functions

After obtaining the dimension formulae of biquadratic spline spaces over hierarchical T-meshes, we naturally consider how to construct their basis functions with some good properties as stated for bilinear basis functions in Section 3.4.

To do so, we fist need to make clear of the topological meaning of $V^+ - E + \delta$.

Definition 5.13 *Given a hierarchical T-mesh $\mathscr{T}$, one can construct a graph $\mathscr{G}$ by keeping all the crossing vertices and the line segments with two endpoints being crossing vertices, and removing all the other vertices and the edges in $\mathscr{T}$. $\mathscr{G}$ is called the* ***crossing-vertex-relationship graph*** *(CVR graph for short) of $\mathscr{T}$.*

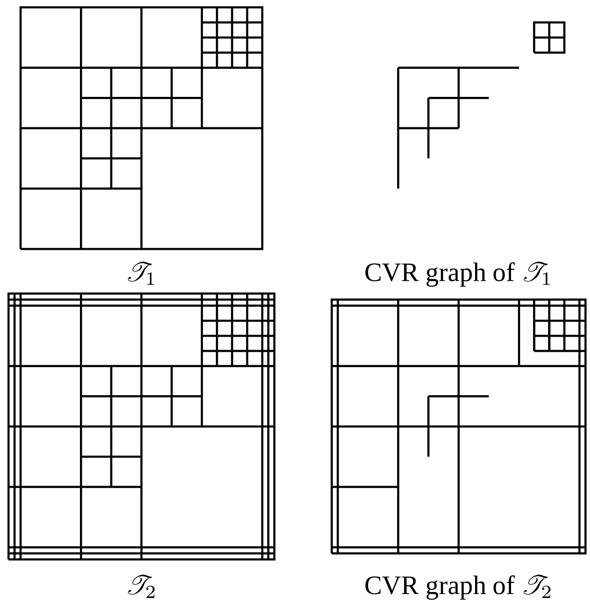


Figure 12: Two examples of CVR graphs.

See Figure 12 for two examples. Here $\mathscr{T}_1$ has an isolated subdivided cell at its right-top part. Hence $\delta = 2$. Its CVR graph has two disconnected parts. $\mathscr{T}_2$ is an extended T-mesh of $\mathscr{T}_1$ with respect to the spline space $\mathbf{S}(2, 2, 1, 1, \mathscr{T}_1)$. $\mathscr{T}_2$ has no isolated subdivided cell. The corresponding CVR graph is connected. The following theorem states the relationship between $V^+ - E + \delta$ in $\mathscr{T}$ and the cell number $F_{\mathscr{G}}$ in $\mathscr{G}$.

Theorem 5.14 *Given a hierarchical T-mesh $\mathscr{T}$ with V^+ crossing vertices, E interior l-edges and $\delta - 1$ isolated subdivided cells, suppose there are $F_{\mathscr{G}}$ cells in its CVR graph $\mathscr{G}$. Then it follows that*

$$V^+ - E + \delta = F_{\mathscr{G}}. \tag{5.10}$$

Proof: Suppose there are $E_{\mathscr{G}}$ edges in $\mathscr{G}$. According to the definition of the CVR graph, it follows that there are δ disconnected parts in $\mathscr{G}$. Hence with the Euler formula, one gets

$$F_{\mathscr{G}} - E_{\mathscr{G}} + V^+ = \delta.$$

Hence in order to prove Equation (5.10), one just needs to show that $2V^+ = E + E_{\mathscr{G}}$. In fact, consider any l-edge ℓ_i with V_i^+ crossing vertices. Then ℓ_i generates $V_i^+ - 1$ edges in $\mathscr{G}$. After running through all the l-edges in $\mathscr{T}$, each of the crossing vertices is met twice. Hence it follows that $2V^+ = E + E_{\mathscr{G}}$, which finishes the proof of the theorem. ∎

Theorem 5.14 hints us that the basis functions of the biquadratic spline space over a hierarchical T-mesh could be constructed around the cells in its CVR graph. Some experiments have been done on this idea, which will be explored in the future.

Furthermore, we expect that CVR graphs will play an important role in dimension analysis and basis function construction of higher degree spline spaces over (hierarchical) T-meshes. For example, we have the following conjecture:

Conjecture 1 *Suppose $\mathscr{T}$ is a hierarchical T-mesh, and its CVR graph is $\mathscr{G}$. As $m, n \geqslant 2$, it follows that*

$$\dim \overline{\mathbf{S}}(m, n, m-1, n-1, \mathscr{T}) = \dim \overline{\mathbf{S}}(m-2, n-2, m-3, n-3, \mathscr{G}),$$

where the spline space $\overline{\mathbf{S}}(m, n, \alpha, \beta, \mathscr{G})$ is defined in a similar way with the spline space over a T-mesh.

Theorem 5.14 states that the conjecture holds as $m = n = 2$. As for $m = n = 3$, we have tried many examples, which support this conjecture as well.

6 Conclusions

In this paper the dimension of bilinear and biquadratic spline spaces over T-meshes are discussed. The basis strategy is by linear space embedding with an operator of mixed partial derivative. We obtained the dimension formula of bilinear spline spaces over general T-meshes, and that of biquadratic spline spaces over hierarchical T-meshes. Only a lower bound of the dimension is build for biquadratic spline spaces over general T-meshes.

In the future, the basis function construction of biquadratic splines spaces and the proposed conjecture will be explored.

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